

# Constraint-Selected and Search-Optimized Families of Daubechies Wavelet Filters Computable by Spectral Factorization

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## Abstract

A unifying algorithm has been developed to systematize the collection of compact Daubechies wavelets computable by spectral factorization of a symmetric positive polynomial. This collection comprises all classes of real and complex orthogonal and biorthogonal wavelet filters with maximal flatness for their minimal length. The main algorithm incorporates spectral factorization of the Daubechies product filter into analysis and synthesis filters. The spectral factors are found for search-optimized families by examining a desired criterion over combinatorial subsets of roots indexed by binary codes, and for constraint-selected families by imposing sufficient constraints on the roots without any optimizing search for an extremal property. Daubechies wavelet filter families have been systematized to include those constraint-selected by the principle of separably disjoint roots, and those search-optimized for time-domain regularity, frequency-domain selectivity, time-frequency uncertainty, and phase nonlinearity. The latter criterion permits construction of the least and most asymmetric and least and most symmetric real and complex orthogonal filters. Biorthogonal symmetric spline and balanced-length filters with linear phase are also computable by these methods. This systematized collection has been developed in the context of a general framework enabling evaluation of the equivalence of constraint-selected and search-optimized families with respect to the filter coefficients and roots and their characteristics. Some of the constraint-selected families have been demonstrated to be equivalent to some of the search-optimized families, thereby obviating the necessity for any search in their computation.

*Key words:* Multirate Filter Design, Wavelets and Filter Banks.

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## 1 Introduction

Since the discovery of compact orthogonal and biorthogonal wavelets by Daubechies, various discussions of the general theory and specific parameteriza-

tions of her wavelets have also been published (*cf.* [2,5,12,16] for literature reviews). These compact Daubechies wavelets, which have the maximal number of vanishing moments for their minimal length, can be implemented as discrete filters that are iterated or auto-convolved to generate approximations of the continuous functions.

The Daubechies wavelet filters can be readily computed via spectral factorization of a symmetric positive polynomial [1]. Significant advantages of the spectral factorization approach include its generalizability to many different classes and families of wavelets, its suitability for easily interpretable visual displays, and thus its practicality in pedagogy. All of the complex orthogonal, real orthogonal, and real biorthogonal families of the Daubechies class computable by spectral factorization and constructed with a single unifying computational algorithm have been studied experimentally in the systematized collection developed by Taswell [10–12,15–17] over a wide range of vanishing moment numbers and filter lengths.

In contrast, angular parameterization methods have usually been demonstrated for wavelets with only one vanishing moment (*i.e.*, less than maximal flatness) and very short lengths [9] with the exception of [13]. But the latter only verified orthogonality and vanishing moment numbers for the filters and did not attempt any search through the angular parametrization space for filters with desirable properties.

These comments highlight one of the essential questions in the development of an algorithm for the design of wavelet filters: How much computational effort should be expended in the construction of a wavelet filter possessing which properties over which range of filter lengths? A basic assumption inherent in the systematized collection of Daubechies wavelets [11,15,17] hypothesizes that the spectral factorization approach affords the most economical generation of wavelet filters with the best variety and combination of properties over the widest range of filter lengths.

The economy of the spectral factorization method in comparison with the angular parameterization method is achieved by the reduced size of the search space for the filter root codes [16] relative to that for the filter coefficient angles [9]. In [16], conjectures were made regarding schemes to enhance the efficiency of the combinatorial search used in the design algorithm. In [17], a new design principle was introduced within a general framework to demonstrate that the search can be completely eliminated for those search-optimized filter families for which equivalence has been demonstrated with constraint-selected filter families. This survey reviews the development of the systematized collection of Daubechies wavelets and summarizes the essential computational methods.

## 2 General Framework

Consider a filter expressed as the complex  $z$ -domain polynomial  $F(z)$  with corresponding vectors for the roots  $\mathbf{z} \equiv [z_j] \in \mathcal{Z}$  and the coefficients  $\mathbf{f} \equiv [f_n] \in \mathcal{F}$ . Associated with  $F(z)$ , assume there exist three parameters, vectors  $\boldsymbol{\gamma} \in \Gamma$ ,  $\boldsymbol{\xi} \in \Xi$ , and scalar  $\lambda \in \Lambda$ , respectively, that index the filter within a set of such filters forming a defined family, specify each indexed filter of the family within a search space, and characterize its properties.

Applying this notation to the orthonormal Daubechies [1] and Rioul [7] wavelets,  $\boldsymbol{\gamma} \equiv [\gamma_1, \gamma_2] = [N, K]$  represents the number  $K$  of vanishing moments for wavelet filters of length  $N = 2K$  and  $N > 2K$ , respectively. For angle space methods [9] to generate orthonormal wavelets,  $\boldsymbol{\xi}$  represents the set of angles that specifies  $\mathbf{f}$  for  $F(z)$ . For binomial space methods [16] to generate Daubechies wavelets,  $\boldsymbol{\xi}$  represents the set of binary codes that specifies  $\mathbf{z}$  for  $F(z)$ . In both cases,  $\lambda$  represents a criterion obtained from an individual property or a weighted combination of properties computed from  $\mathbf{z}$  and/or  $\mathbf{f}$  (such as the filter's time-domain regularity [14], phase nonlinearity [16], *et c.*) that characterizes  $F(z)$ .

Thus,  $\boldsymbol{\gamma}$  and  $\boldsymbol{\xi}$  determine  $F(z)$  and then  $F(z)$  determines  $\lambda$  with the mapping of spaces  $\Gamma \times \Xi \mapsto \mathcal{F} \times \mathcal{Z} \mapsto \Lambda$ . The parameters  $\boldsymbol{\gamma}$  and  $\boldsymbol{\xi}$  that determine  $F(z)$  are called the *indexing parameter* and *specification parameter*, respectively. The parameter  $\lambda$  that is determined by  $F(z)$  is called the *characterization parameter*. If  $\lambda$  represents an individual property (rather than weighted combination of properties), then  $\lambda$  is also termed a *characteristic property* of  $F(z)$ .

### 2.1 Existence and Uniqueness

Given a defined filter family  $\{F_{\boldsymbol{\gamma}}(z)\}$  indexed by  $\boldsymbol{\gamma}$ , assume for fixed  $\boldsymbol{\gamma}$  that a finite sequence of filters  $F_{\boldsymbol{\gamma},i}(z)$  indexed by  $i$  can be generated by and evaluated for corresponding sequences, respectively, of specification parameters  $\boldsymbol{\xi}_i$  and characterization parameters  $\lambda_i$ . If  $\Xi$  is an unbounded or continuous space, then it can be appropriately bounded and discretized to permit a countably finite sequence  $\boldsymbol{\xi}_i$ .

Assuming restriction to a countably finite space  $\Xi$ , then the corresponding spaces  $\mathcal{F} \times \mathcal{Z}$  and  $\Lambda$  are also countably finite. Further assuming a one-to-one invertible mapping and uniqueness of the elements  $\lambda_i \in \Lambda$  (achieved if necessary by the use of “tie-breaker” rules for the definition of the characterization parameter  $\lambda$ ), then finite countability of unique elements for an invertible mapping implies that it is feasible to search for both elements  $\underline{\lambda} \equiv \min_i \lambda_i$  and  $\bar{\lambda} \equiv \max_i \lambda_i$  in the range and select the corresponding filters  $F_{\boldsymbol{\gamma},i}(z)$  in the

domain.

## 2.2 Definitions and Inferences

A filter  $F(z)$  is called *extremal* if it can be shown to possess a characterization parameter attaining an extreme manifested by either  $\underline{\lambda}$  or  $\bar{\lambda}$ . A filter  $F(z)$  is called *search optimized* if it is generated by an algorithm that optimizes  $\lambda \in \Lambda$  with an exhaustive search to ensure identification of either  $\underline{\lambda}$  or  $\bar{\lambda}$ . A filter  $F(z)$  is called *constraint selected* if it is generated by an algorithm that specifies sufficient constraints on  $\boldsymbol{\xi}$ ,  $\mathbf{f}$ , or  $\mathbf{z}$  to ensure uniqueness of  $F(z)$  and selection of  $F(z)$  without a search. An indexed set of filters  $\{F_\gamma(z)\} \equiv \{F(z; \gamma) : \gamma \in \Gamma\}$  is called a *family* if all members of the set are generated by the *same* algorithm, a function  $g(\boldsymbol{\xi}; \gamma)$ ,  $g(\mathbf{f}; \gamma)$ , or  $g(\mathbf{z}; \gamma)$ , subject to the control of the indexing parameter  $\gamma$ .

Two different filter families  $\{F_\gamma(z)\}$  and  $\{F'_\gamma(z)\}$  generated by two different algorithms  $g(\cdot; \gamma)$  and  $g'(\cdot; \gamma)$  are  $\mathcal{F}$ -*equivalent*, or equivalent with respect to (w.r.t.) the filter coefficient space  $\mathcal{F}$ , if  $\|\mathbf{f}_\gamma - \mathbf{f}'_\gamma\| < \tau$  for all  $\gamma \in \Gamma$  with given error tolerance  $\tau(\mathcal{F})$ . Analogously,  $\{F_\gamma(z)\}$  and  $\{F'_\gamma(z)\}$  are  $\mathcal{Z}$ -*equivalent*, or equivalent w.r.t. the filter root space  $\mathcal{Z}$ , if  $\|\mathbf{z}_\gamma - \mathbf{z}'_\gamma\| < \tau$  for all  $\gamma \in \Gamma$  with given error tolerance  $\tau(\mathcal{Z})$ . Finally, they are  $\Lambda$ -*equivalent*, or equivalent w.r.t. the characterization parameter space  $\Lambda$ , if  $|\lambda_\gamma - \lambda'_\gamma| < \tau$  for all  $\gamma \in \Gamma$  with given error tolerance  $\tau(\Lambda)$ .

A search-optimized filter is necessarily an extremal filter, whereas a constraint-selected filter may or may not be an extremal filter. If a constraint-selected filter can be shown to be equivalent to a search-optimized filter, then the constraint-selected filter is also an extremal filter. Both  $\mathcal{F}$ -equivalence and  $\mathcal{Z}$ -equivalence of two different filter families imply  $\Lambda$ -equivalence, but the converse is not true.

## 3 Daubechies Polynomials

The generation of Daubechies wavelet filter families computable by spectral factorization of the Daubechies polynomials requires a separate algorithm for computing the roots of the product filter

$$\mathcal{P}_D(z) = (z + 1)^{2(D+1)} \mathcal{Q}_D(z) \quad (1)$$

or its related form the quotient filter

$$\mathcal{Q}_D(z) = (z + 1)^{-2(D+1)} \mathcal{P}_D(z) \quad (2)$$

which is a Laurent polynomial of degree  $d_2 = D$  with  $2D$  roots. Both forms are indexed by the integer parameter  $D \geq 0$ .

Consider mappings  $x \rightarrow y \rightarrow z$  between three planes in the complex variables  $x$ ,  $y$ , and  $z$ . Use the  $x$  plane to find the roots of the conditioned polynomial  $\mathcal{C}_D(x)$ , map to the  $y$  plane for the roots of the binomial polynomial  $\mathcal{B}_D(y)$ , and map again to the  $z$  plane for the roots of the quotient polynomial  $\mathcal{Q}_D(z)$ . All three polynomials  $\mathcal{C}_D(x)$ ,  $\mathcal{B}_D(y)$ , and  $\mathcal{Q}_D(z)$  are considered related forms of  $\mathcal{P}_D(z)$  called the conditioned, binomial, and quotient forms, respectively.

The quotient form  $\mathcal{Q}_D(z)$  derives simply from division of the product form  $\mathcal{P}_D(z)$  by all of its roots at  $z = -1$ . The binomial form ([2, Eq. 6.1.12], [8, Eq. 1], [3, Eq. 1.7])

$$\mathcal{B}_D(y) = \sum_{i=0}^D \binom{D+i}{i} y^i \quad (3)$$

derives from the binomial series for  $(1-y)^{-(D+1)}$  truncated at  $D+1$  terms. These forms can be related through conformal mappings (see below).

To improve the numerical conditioning of the root finding problem for the roots  $y_i$  of  $\mathcal{B}_D(y)$ , Shen and Strang [8] recommended the change of variables  $x = \kappa y$  with  $\kappa = 4$ , while Goodman *et al.* [3] recommended the change of variables  $x = 1/y$ . Incorporating both transformations with  $x = 1/(\kappa y)$ , then

$$\begin{aligned} \mathcal{B}_D(y) &= \sum_{i=0}^D \binom{D+i}{i} y^i \\ &= (\kappa y)^D \sum_{i=0}^D \kappa^{-i} \binom{D+i}{i} (\kappa y)^{i-D} \\ &= x^{-D} \mathcal{C}_D(x) \end{aligned}$$

yields the conditioned form

$$\mathcal{C}_D(x) = \sum_{i=0}^D \kappa^{-i} \binom{D+i}{i} x^{D-i}. \quad (4)$$

Now obtain the  $D$  roots  $x_i$  of  $\mathcal{C}_D(x)$  by computing the eigenvalues of the companion matrix. Then the  $D$  roots  $y_i$  of the binomial form  $\mathcal{B}_D(y)$  can be calculated simply as  $y_i = 1/(\kappa x_i)$ .

With another change of variables  $z + z^{-1} = 2 - 4y$  as described by Daubechies [1,2], map the binomial form  $\mathcal{B}_D(y)$ , a regular polynomial with  $D$  roots, to the quotient form  $\mathcal{Q}_D(z)$ , a Laurent polynomial with  $2D$  roots. Given the Joukowski transformations [4, vol 1, pg 197, 223]

$$w = f(z) = (z + z^{-1})/2 \quad (5)$$

$$z = f^{-1}(w) = w \pm \sqrt{w^2 - 1} \quad (6)$$

and the affine transformations

$$y = g(w) = (1 - w)/2 \quad (7)$$

$$w = g^{-1}(y) = 1 - 2y, \quad (8)$$

then the composite mappings<sup>1</sup> yield the explicit solutions

$$y = g(f(z)) = (1 - (z + z^{-1})/2)/2 \quad (9)$$

$$z = f^{-1}(g^{-1}(y)) = 1 - 2y \pm \sqrt{(1 - 2y)^2 - 1}. \quad (10)$$

The latter equation yields a doubly-valued solution with the reciprocal pair  $\{z, z^{-1}\}$ . When the pairs are regrouped as complex quadruplets  $\{z, z^{-1}, \bar{z}, \bar{z}^{-1}\}$  and factors  $\mathcal{U}(z; z_i) \equiv (z - z_i)(z - z_i^{-1})(z - \bar{z}_i)(z - \bar{z}_i^{-1})$  with any real duplets  $\{r, r^{-1}\}$  and factors  $\mathcal{V}(z; r_j) \equiv (z - r_j)(z - r_j^{-1})$ , the Daubechies product polynomial  $\mathcal{P}_D(z)$  expressed in regular form can be factored as

$$\mathcal{P}_D(z) = (z + 1)^{2(D+1)} \prod_{i=1}^{n^{\text{cq}}} \mathcal{U}(z; z_i) \prod_{j=1}^{n^{\text{rd}}} \mathcal{V}(z; r_j) \quad (11)$$

where  $n^{\text{cq}} = \lfloor D/2 \rfloor$  and  $n^{\text{rd}} = D \bmod 2$ . For further details on the numerical performance of these methods, refer to [12,16].

## 4 Spectral Factorization Rules

For an arbitrary polynomial  $\mathcal{F}(z)$  with length  $N$  coefficients, there are  $N - 1$  roots of which  $0 \leq K \leq N - 1$  may be at  $z = -1$ . When considering spectral factorization, the product filter polynomial  $\mathcal{P}_D(z)$  with  $N_p = 4D + 3$  coefficients and  $K_p = 2D + 2$  roots at  $z = -1$  is factored into the analysis and synthesis filter polynomials  $\mathcal{A}(z)$  and  $\mathcal{S}(z)$  with  $N_a$  and  $N_s$  coefficients, and  $K_a$  and  $K_s$  roots at  $z = -1$ , respectively. This factorization yields the constraints

$$N_p = N_a + N_s - 1 \quad (12)$$

$$K_p = K_a + K_s \quad (13)$$

<sup>1</sup> Unlike other sections where  $f$  and  $g$  may denote filters or arbitrary functions, here  $f$  and  $g$  denote functions that are conformal maps in the complex domain.

on the lengths of the three filters and their roots at  $z = -1$ . Each family of filters described in subsequent sections has been named with an identifying acronym followed by  $(N; K)$  in the orthogonal cases for which

$$N = N_a = N_s \quad (14)$$

$$K = K_a = K_s \quad (15)$$

is required, and by  $(N_a, N_s; K_a, K_s)$  in the biorthogonal cases for which

$$N_a = K_a + 4n_a^{\text{cq}} + 2n_a^{\text{rd}} + 1 \quad (16)$$

$$N_s = K_s + 4n_s^{\text{cq}} + 2n_s^{\text{rd}} + 1 \quad (17)$$

$$N_p = 2K_p - 1 \quad (18)$$

is required. Here  $n_a^{\text{cq}}, n_s^{\text{cq}}, n_a^{\text{rd}}$ , and  $n_s^{\text{rd}}$  are the numbers of complex quadruplet factors  $\mathcal{U}(z; z_i)$  and real duplet factors  $\mathcal{V}(z; r_j)$  for each of  $\mathcal{A}(z)$  and  $\mathcal{S}(z)$ . Both  $n^{\text{cq}}$  and  $n^{\text{rd}}$  may be whole or half integer. In the latter case, half of a complex quadruplet and half of a complex duplet denote, respectively, a complex duplet and a real singlet.

For  $K_a$  and  $K_s$  necessarily both odd or both even, then  $K_p$  is always even and  $K = K_p/2$  a whole integer determines  $n_p^{\text{cq}} = n_a^{\text{cq}} + n_s^{\text{cq}}$  and  $n_p^{\text{rd}} = n_a^{\text{rd}} + n_s^{\text{rd}}$  according to  $n_p^{\text{cq}} = \lfloor (K - 1)/2 \rfloor$  and  $n_p^{\text{rd}} = (K - 1) \bmod 2$ . If  $K_a$  and  $K_s$  are given, then  $K_p$  and  $K$  yield  $n_p^{\text{cq}}$  and  $n_p^{\text{rd}}$  split into  $\{n_a^{\text{cq}}, n_s^{\text{cq}}\}$  and  $\{n_a^{\text{rd}}, n_s^{\text{rd}}\}$  and the roots are factored accordingly. For real coefficients, a root  $z$  must be paired with its conjugate  $\bar{z}$ . For symmetric coefficients, a root  $z$  must be paired with its reciprocal  $z^{-1}$ . For 2-shift orthogonal coefficients, a root  $z$  must be *separated* from its conjugate reciprocal  $\bar{z}^{-1}$ .

Thus, in the real biorthogonal symmetric case, each complex quadruplet  $\mathcal{U}(z; z_i)$  and real duplet  $\mathcal{V}(z; r_j)$  must be assigned in its entirety to either  $\mathcal{A}(z)$  or  $\mathcal{S}(z)$ . In the real orthogonal case, each complex quadruplet is split into two conjugate duplets  $(z - z_i)(z - \bar{z}_i)$  and  $(z - z_i^{-1})(z - \bar{z}_i^{-1})$ , while each real duplet is split into two singlets  $(z - r_j)$  and  $(z - r_j^{-1})$ , with one factor assigned to  $\mathcal{A}(z)$  and the other to  $\mathcal{S}(z)$ . The complex orthogonal case is analogous to the real orthogonal case except that the complex quadruplets are split into reciprocal duplets  $(z - z_i)(z - z_i^{-1})$  and  $(z - \bar{z}_i)(z - \bar{z}_i^{-1})$  instead of conjugate duplets. The complex orthogonal symmetric case requires use of complex quadruplets without real duplets.

All orthogonal cases require  $K = K_a = K_s = K_p/2$ ,  $n_a^{\text{cq}} = n_s^{\text{cq}} = n_p^{\text{cq}}/2$ , and  $n_a^{\text{rd}} = n_s^{\text{rd}} = n_p^{\text{rd}}/2$  with  $N = N_a = N_s = 2K$ . Note that  $n_p^{\text{rd}}$  can only equal 0 or 1. Therefore, in biorthogonal cases, either  $\{n_a^{\text{rd}} = 0, n_s^{\text{rd}} = 1\}$  or  $\{n_a^{\text{rd}} = 1, n_s^{\text{rd}} = 0\}$ . However, in orthogonal cases, either  $\{n_a^{\text{rd}} = n_s^{\text{rd}} = 0\}$  or  $\{n_a^{\text{rd}} = n_s^{\text{rd}} = 1/2\}$  with  $1/2$  of a duplet denoting a singlet. For all real orthogonal cases

as well as those complex orthogonal cases not involving symmetry criteria,  $K$  can be any positive integer. For the complex orthogonal least-asymmetric and most-asymmetric cases,  $K$  must be a positive even integer. For the complex orthogonal least-symmetric and most-symmetric cases,  $K$  must be a positive odd integer.

For the real biorthogonal symmetric cases,  $K_a$  and  $K_s$  must be both odd or both even. In the biorthogonal symmetric spline case, all additional roots (other than those at  $z = -1$  with assignment determined by  $K_a$  and  $K_s$ ) are assigned to the analysis filter leaving the synthesis filter as the spline filter. All other biorthogonal symmetric cases incorporate a root assignment constraint that balances the lengths of the analysis and synthesis filters such that  $N_a \approx N_s$  as much as possible. For  $K_a = 2i - 1$  and  $K_s = 2j - 1$  both odd with  $i, j \in \{1, 2, 3, \dots\}$ , balancing of equal filter lengths is possible. In fact, requiring *both*  $K_a = K_s$  and  $N_a = N_s$  is also possible when  $N = N_a = N_s = 2K$  with  $K = K_a = K_s$  for  $\{K = 1 + 4k \mid k = 1, 2, 3 \dots\}$ . However, for  $K_a = 2i$  and  $K_s = 2j$  both even, equal balancing of filter lengths  $N_a$  and  $N_s$  is not possible. The additional unbalanced roots are assigned to the analysis filter such that  $N_a > N_s$  leaving the synthesis filter as the shorter filter.

## 5 Daubechies Wavelet Filter Families

All filter families surveyed here are named, defined, and generated according to the conventions, notation, and methods established in [15,16] for the systematized collection of wavelet filters computable by spectral factorization of the Daubechies polynomial. However, one of the original families, named DROLD in [15], was renamed DROMD in [17] in order to achieve consistency with the more recent collection of families introduced in [17]. All of the acronyms used for the filter family names abbreviate ‘D’ for Daubechies as the first character, ‘C’ or ‘R’ for complex or real as the second character, ‘O’ or ‘B’ for orthogonal or biorthogonal as the third character, and then two additional characters denoting an additional description to distinguish each family from the others.

### 5.1 Constraint-Selected Families

In addition to the spectral factorization rules (Section 4) imposing the necessary constraints for complex orthogonality, real orthogonality, and real biorthogonality, the least and most disjoint families are defined according to constraints derived from the principle of separably disjoint root sets in the complex  $z$ -domain. Consider only the roots of the quotient polynomial  $Q(z)$  (Equation 2) and split this set of roots into two sets of roots  $\{z_k^a\}$  and  $\{z_l^s\}$  for the analysis

and synthesis filters  $\mathcal{A}(z)$  and  $\mathcal{S}(z)$ .

These root sets from  $\mathcal{Q}(z)$  must be disjoint with

$$\emptyset = \{z_k^a\} \cap \{z_l^s\} \quad (19)$$

(because common roots at  $z = -1$  for both  $\mathcal{A}(z)$  and  $\mathcal{S}(z)$  from  $\mathcal{P}(z)$  have been excluded from consideration). Now let  $\{\mathcal{C}_i^a\}$  and  $\{\mathcal{C}_j^s\}$  denote finite collections of open convex regions with the largest area domains that do not intersect yet still cover the sets  $\{z_k^a\}$  and  $\{z_l^s\}$ , respectively. More precisely,

$$\cup_k z_k^a \subset \cup_i \mathcal{C}_i^a \quad (20)$$

$$\cup_l z_l^s \subset \cup_j \mathcal{C}_j^s \quad (21)$$

$$\emptyset = \cap_i \mathcal{C}_i^a \quad (22)$$

$$\emptyset = \cap_j \mathcal{C}_j^s \quad (23)$$

$$\emptyset = (\cup_i \mathcal{C}_i^a) \cap (\cup_j \mathcal{C}_j^s). \quad (24)$$

Finally, let  $C$  denote the cardinality of the set

$$\{\mathcal{C}_i^a : i = 1, \dots, I; \mathcal{C}_j^s : j = 1, \dots, J\} \quad (25)$$

as measured by the number  $C = I + J$  of regions covering all the roots of  $\mathcal{Q}(z)$ . Then root sets  $\{z_k^a\}$  and  $\{z_l^s\}$  are called least and most disjoint if  $C$  is, respectively, the maximum or minimum possible subject to the constraints of the spectral factorization rules imposed.

Table 1

Filter Designs for Some Constraint-Selected Families with Roots  $z_j = r_j e^{i\theta_j}$

Acronym	$\mathcal{Q}(z) \rightarrow \mathcal{A}(z)$	$\mathcal{Q}(z) \rightarrow \mathcal{S}(z)$
DCOMD	$\{(z_j, z_j^{-1}) : (r_j < 1) \wedge (\theta_j \geq 0)\}$	$\{(z_j, z_j^{-1}) : (r_j > 1) \wedge (\theta_j \leq 0)\}$
DROMD	$\{(z_j, \bar{z}_j) : r_j < 1\}$	$\{(z_j, \bar{z}_j) : r_j > 1\}$
DRBMD	$\{(z_j, \bar{z}_j, z_j^{-1}, \bar{z}_j^{-1}) : \theta_j < \theta^*\}$	$\{(z_j, \bar{z}_j, z_j^{-1}, \bar{z}_j^{-1}) : \theta_j > \theta^*\}$
DRBSS	$\{(z_j, \bar{z}_j, z_j^{-1}, \bar{z}_j^{-1})\}$	$\emptyset$

Table 1 summarizes the spectral factorizations for the DCOMD, DROMD, and DRBMD filter families designed with most disjoint (MD) root sets. The factorizations for the DCOLD, DROLD, and DRBLD filters designed with least disjoint (LD) root sets cannot be summarized as concisely. However, the corresponding algorithms order the roots by angle and impose the maximum number of alternations for the assignments in the split to  $\mathcal{A}(z)$  and  $\mathcal{S}(z)$ . The algorithm for DRBLD was also modified to devise another family called DRBRD with regular disjoint (RD) root sets. For comparison, Table 1 also includes the spectral factorization for the DRBSS family with symmetric spline (SS) root sets.

## 5.2 Search-Optimized Families

Numerical estimates of defined filter characterization parameters  $\lambda$  are used as selection criteria for all other families subjected to optimization in combinatorial searches of the root sets. These criteria [14] include the phase nonlinearity  $\text{pnl}(\mathcal{A})$ , time-domain regularity  $\text{tdr}(\mathcal{A})$ , frequency-domain selectivity  $\text{fds}(\mathcal{A})$ , and time-frequency uncertainty  $\text{tfu}(\mathcal{A})$ . Most of the orthogonal families are defined by  $\text{pnl}(\mathcal{A})$  selecting for varying degrees of asymmetry or symmetry. Work reported in [11,12,15] was later revised in [16] by the shift of the integration interval for  $\text{pnl}(\mathcal{A})$  from  $[0, 2\pi]$  to  $[-\pi, \pi]$  and by the use of  $\text{pnl}(\mathcal{A})$  as a “tie-breaker” criterion for families selected by the other criteria. These revisions now insure unique criterion values for each root set examined in the combinatorial search (which can be performed ignoring binary complements for orthogonal families).

Minimizing or maximizing  $\text{pnl}(\mathcal{A})$  for real filters defines DROLA and DROMA, respectively, the least asymmetric (LA) and most asymmetric (MA) families. If the parity of  $K$  is ignored, then minimizing or maximizing  $\text{pnl}(\mathcal{A})$  for complex filters defines DCOLN and DCOMN, respectively, the least nonlinear (LN) and most nonlinear (MN) families. Phase nonlinearity does not exist and cannot be used for the real biorthogonal families all of which are symmetric. Therefore, one of the other characterization parameters must be used as an optimization criterion. Also, these biorthogonal families are subjected to the length constraints determined by the principle of maximally balancing the filter lengths for both  $\mathcal{A}(z)$  and  $\mathcal{S}(z)$ .

For all but several of the search-optimized families, the selection criterion is optimized for  $\mathcal{A}(z)$ . The exceptions are the DRBBR, DRBBS, and DRBBU families with balanced regular (BR), balanced selective (BS), and balanced uncertain (BU) root sets. Instead, the selection criterion is optimized for both  $\mathcal{A}(z)$  and  $\mathcal{S}(z)$  by maximizing a balancing measure  $B$  defined as

$$B(\lambda(\cdot), \mathcal{A}, \mathcal{S}) = \left| \frac{\lambda(\mathcal{A}) + \lambda(\mathcal{S})}{\lambda(\mathcal{A}) - \lambda(\mathcal{S})} \right| \quad (26)$$

where  $\lambda(\cdot)$  is either  $\text{tdr}(\cdot)$ ,  $\text{fds}(\cdot)$ , or  $\text{tfu}(\cdot)$ , respectively, for DRBBR, DRBBS, and DRBBU.

Table 2 summarizes filter designs for some of the search-optimized families. The index constraints tabulated are those required to generate the defined family. However, for purposes of comparison between families in tables and figures, the definitions for all orthogonal families have been extended to begin at  $K = 1$ . For example, DCOLN(6;3) is complex as expected, but DCOLN(4;2) and DCOLN(2;1) are real. Also, note that the DCOLN family is the union of the even-indexed DCOLA and odd-indexed DCOMS families, while the

Table 2  
Filter Designs for Some Search-Optimized Families

Real Biorthogonal	Description	Index Constraint	Optimization
DRBLU	Least Uncertain	even $(K_a + K_s)$	$\min \text{tfu}(\mathcal{A})$
DRBMS	Most Selective	even $(K_a + K_s)$	$\max \text{fds}(\mathcal{A})$
DRBMR	Most Regular	even $(K_a + K_s)$	$\max \text{tdr}(\mathcal{A})$
DRBBR	Balanced Regular	even $(K_a + K_s)$	$\max B(\text{tdr}(\cdot), \mathcal{A}, \mathcal{S})$

Real Orthogonal	Description	Constraint	Optimization
DROLU	Least Uncertain	$K \geq 1$	$\min \text{tfu}(\mathcal{A})$
DROMR	Most Regular	$K \geq 1$	$\max \text{tdr}(\mathcal{A})$
DROLA	Least Asymmetric	$K \geq 1$	$\min \text{pnl}(\mathcal{A})$
DROMA	Most Asymmetric	$K \geq 1$	$\max \text{pnl}(\mathcal{A})$

Complex Orthogonal	Description	Constraint	Optimization
DCOLU	Least Uncertain	$K \geq 3$	$\min \text{tfu}(\mathcal{A})$
DCOMR	Most Regular	$K \geq 3$	$\max \text{tdr}(\mathcal{A})$
DCOLS	Least Symmetric	odd $K \geq 3$	$\max \text{pnl}(\mathcal{A})$
DCOMS	Most Symmetric	odd $K \geq 3$	$\min \text{pnl}(\mathcal{A})$
DCOLA	Least Asymmetric	even $K \geq 4$	$\min \text{pnl}(\mathcal{A})$
DCOMA	Most Asymmetric	even $K \geq 4$	$\max \text{pnl}(\mathcal{A})$
DCOLN	Least Nonlinear	$K \geq 3$	$\min \text{pnl}(\mathcal{A})$
DCOMN	Most Nonlinear	$K \geq 3$	$\max \text{pnl}(\mathcal{A})$

DCOMN family is the union of the even-indexed DCOMA and odd-indexed DCOLS families. Complete details for the algorithms to compute each of the various selection criteria can be found elsewhere [12,14].

## 6 Unifying Algorithm

All filter families of the systematized collection of Daubechies wavelet filters [12,16] are generated by the spectral factorization and selection of root sets (with either the predetermined constraints or the optimizing combinatorial

search) incorporated in the following algorithm:

- (1) Input the identifying name **FilterName** for the family of filters and the indexing design parameters  $K_a$  and  $K_s$ .
- (2) Compute the numbers  $K_p = K_a + K_s$ ,  $D = K_p/2 - 1$ ,  $n_p^{\text{cq}} = \lfloor D/2 \rfloor$ , and  $n_p^{\text{rd}} = D \bmod 2$ .
- (3) Compute the  $n_p^{\text{cq}}$  sets of complex quadruplet roots and the  $n_p^{\text{rd}}$  sets of real duplet roots of the quotient filter  $\mathcal{Q}_D(z)$ .
- (4) Access the factorization and selection rules that define the family of filters named **FilterName**.
- (5) Apply the rules to  $\{n_p^{\text{cq}}, n_p^{\text{rd}}\}$  for the **FilterName** filter pair indexed by  $\{K_a, K_s\}$  and compute the splitting number pairs  $\{n_a^{\text{cq}}, n_s^{\text{cq}}\}$  and  $\{n_a^{\text{rd}}, n_s^{\text{rd}}\}$ .
- (6) If **FilterName** is a constraint-selected family, apply the rules to select the  $4n_a^{\text{cq}} + 2n_a^{\text{rd}}$  roots for  $\mathcal{A}(z)$  and the  $4n_s^{\text{cq}} + 2n_s^{\text{rd}}$  roots for  $\mathcal{S}(z)$  and jump to Step 11.
- (7) Sort the roots in an order convenient for the class of splitting appropriate to the type of filter. All roots of a complex quadruplet should be adjacent with duplets of the quadruplet subsorted according to conjugates or reciprocals depending on the filter type. Assign binary coded labels 0 and 1 to the first and second duplet of each quadruplet. Analogously, assign binary codes to the first and second singlet of the real reciprocal duplet if present. If biorthogonal, assign binary coded labels 0 or 1 to each of the entire quadruplets and duplets.
- (8) Generate the possible binomial subsets for these binary codes [6] subject to the imposed factorization rules and splitting numbers. For orthogonal filters, there are a total of  $n_a^{\text{cq}} + n_a^{\text{rd}}$  binary selections without constraint on the bit sum, and thus  $2^{n_a^{\text{cq}} + n_a^{\text{rd}} - 1}$  binomial subsets ignoring complements. For biorthogonal filters, there are a total of  $n_p^{\text{cq}}$  binary selections with bit sum constrained to  $n_a^{\text{cq}}$ , and thus  $\binom{n_p^{\text{cq}}}{n_a^{\text{cq}}}$  binomial subsets.
- (9) For each root subset selected by the binomial subset codes, characterize the corresponding filter by the optimization criterion appropriate for the **FilterName** family. These optimization criteria may be any of the numerically estimated characterization parameters  $\lambda$  computed from the roots  $\mathbf{z}$  or the coefficients  $\mathbf{f}$ .
- (10) Search all root subsets to find the one with the optimal value of the desired criterion. If necessary, apply the “tie-breaker” criterion.
- (11) Include the  $K_a + K_s$  required roots at  $z = -1$  with  $K_a$  for the optimal subset of roots intended for the analysis factor  $\mathcal{A}(z)$  and with  $K_s$  for the complementary subset intended for the synthesis factor  $\mathcal{S}(z)$  and compute the filter coefficients.
- (12) If **FilterName** is an orthogonal search-optimized family, compare the selected (primary) subset of filter roots and coefficients with its complementary subset to choose the one with minimax group delay over the interval  $\omega \in [0, \pi]$  as the subset for  $\mathcal{A}(z)$ . If **FilterName** is a biorthogonal

search-optimized family, compare the primary and complementary subsets only if  $K_a = K_s$ ,  $n_a^{cq} = n_s^{cq}$ , and  $n_a^{rd} = 0 = n_s^{rd}$  in order to choose the one with the defining criterion optimized for  $\mathcal{A}(z)$ .

(13) Output roots  $\mathbf{z}$  and coefficients  $\mathbf{f}$  for each of  $\mathcal{A}(z)$  and  $\mathcal{S}(z)$ .

For search-optimized families, full searches of all possible combinatorial subsets should be performed for a sufficient number of values of  $K$  indexing the filter family's members in order to infer the appropriate pattern of binary codes with bit sums characterizing the family. Using such a pattern permits successful *partial* rather than *full* combinatorial searches. These partial searches provide significant reduction in computational complexity convenient for larger values of  $K$ , for example, for searches with  $K > 30$  computed on desktop workstations current in 1999.

## 7 Examples and Comparisons

Figure 1 displays spectral factorizations for each of the least and most disjoint filter families at  $K_a = K_s = 16$  for  $D = 15$ . Roots for  $\mathcal{A}(z)$  and  $\mathcal{S}(z)$  are marked with “o” and “x”, respectively. As an example of the principle of minimizing and maximizing the cardinality  $C$ , observe that  $C = 3$  for DRBMD and  $C = 13$  for DRBLD. Note that  $C \neq 2$  for DRBMD because convexity is required for each of the non-intersecting covering regions, and  $C \neq 26$  for DRBLD because the largest area possible is required for each of the regions. Figure 2 displays the wavelets corresponding to  $\mathcal{A}(z)$  for the six examples in Figure 1. Both the real parts (solid lines) and imaginary parts (dotted lines) are shown for complex scalets and wavelets.

All filters of all families were demonstrated to meet or surpass requirements for orthogonality, biorthogonality, and reconstruction when tested [14] in 2-band wavelet filter banks. In general, reconstruction errors ranged from “perfect” at  $\mathcal{O}(10^{-16})$  to “near-perfect” at  $\mathcal{O}(10^{-8})$  as  $K$  ranged from  $K = 1$  to  $K = 24$  for both orthogonal and biorthogonal classes. All search-optimized filter families were observed to have the optimal values of their defining selection criterion when compared to the other families.

Figures 3–6 display values of various characteristic properties for the filter families. The families are listed in the legends sorted in order of the properties' median values for  $\mathcal{A}(z)$  over the range of the indexing parameter. These figures and the corresponding numerical values in tables can be examined to assess  $\Lambda$ -equivalence. Refer to [12,16] for a complete catalogue of all results for all of the filter families with both numerical tables of parameter estimates and graphical displays of the filters in the time, frequency, and  $z$  domains.

Although named distinctly because of their different computational algorithms, there are several pairs of filter families which should ideally be  $\mathcal{F}$ -,  $\mathcal{Z}$ - and  $\Lambda$ -equivalent. These pairs provide a test for verifying computational methods. The DROMD and DROMA families should be equivalent real families, while the DCOMD and DCOMN families should be equivalent complex families. Numerical experiments have confirmed these expected results. All constraint-selected families have been compared with the search-optimized families for  $K_a = K_s = 1, \dots, 24$ . Each member of the following sets of filter families have been demonstrated to be  $\mathcal{F}$ -equivalent to the other members of the set with  $\tau(\mathcal{F})$  at machine precision:  $\{\text{DRBMD}, \text{DRBMU}, \text{DRBLS}, \text{DRBLR}\}$ ,  $\{\text{DRBRD}, \text{DRBMR}\}$ ,  $\{\text{DROMD}, \text{DROMA}\}$ , and  $\{\text{DCOMD}, \text{DCOMN}\}$ .

Figures 3 and 4 present visually dramatic contrasting examples of the presence and absence of  $\Lambda$ -equivalence, respectively, for the orthogonal and biorthogonal families with regard to the property of time-domain regularity. Examination of these figures reveals that of those displayed, all of the orthogonal families, but none of the biorthogonal families, are  $\Lambda$ -equivalent with  $\tau(\Lambda) < 0.2$  for time-domain regularity. Figures 5 and 6 demonstrate that  $\{\text{DROLD}, \text{DROLU}\}$  and  $\{\text{DROLD}, \text{DROLA}\}$  are each  $\Lambda$ -equivalent pairs of orthogonal families, respectively, with regard to time-frequency uncertainty and phase nonlinearity. Analogous results for biorthogonal families have shown that  $\{\text{DRBMR}, \text{DRBLU}\}$  is a  $\Lambda$ -equivalent pair with regard to time-frequency uncertainty for  $\mathcal{A}(z)$ , but there is no such pair with regard to frequency-domain selectivity. Note that since the pair  $\{\text{DRBRD}, \text{DRBMR}\}$  is  $\mathcal{F}$ -equivalent, then the pair  $\{\text{DRBRD}, \text{DRBMR}\}$  is  $\Lambda$ -equivalent with regard to time-domain regularity and the pair  $\{\text{DRBRD}, \text{DRBLU}\}$  is  $\Lambda$ -equivalent with regard to time-frequency uncertainty.

## 8 Discussion

An algorithm has been developed to unify all of the diverse families of real and complex orthogonal and biorthogonal Daubechies wavelets. This automated algorithm is valid for any order  $K$  of wavelet and insures that the same consistent choice of roots is always made in the computation of the filter coefficients. It is also sufficiently flexible and extensible that it can be generalized to select roots for filters designed by criteria other than those that already comprise the systematized collection of Daubechies wavelets [11,15,17].

Systematizing a collection of filters with a mechanism both for generating and evaluating the filters enables the development of filter catalogues with tables of numerical parameter estimates characterizing their properties. Providing estimates for a variety of characteristics in both time and frequency domains, rather than just the optimized characteristic, constitutes an important as-

pect of these tables which enhances their utility. Use of these catalogues as a resource enables the investigator to choose an available filter with the desirable characteristics most appropriate to his research problem or development application.

The systematized collection of Daubechies wavelets has been developed within the context of a general filter design framework consisting of indexing parameters  $\gamma \in \Gamma$ , specification parameters  $\xi \in \Xi$ , filter coefficients  $\mathbf{f} \in \mathcal{F}$ , filter roots  $\mathbf{z} \in \mathcal{Z}$ , characterization parameters  $\lambda \in \Lambda$ , their corresponding spaces, and the mappings between the spaces. Within this framework, definitions have been introduced for filter families that are either search optimized or constraint selected, for the equivalence of families, and for new design principles based on disjoint root sets and filter characteristic properties.

Several pairs of both  $\mathcal{F}$ -equivalence and  $\Lambda$ -equivalence have been demonstrated for both orthogonal and biorthogonal classes of filter families. If  $\Lambda$ -equivalence exists between a constraint-selected family and a search-optimized family with respect to a particular characterization parameter  $\lambda$  as an extremal property, then the constraint-selected family can be used to replace the search-optimized family, and thus to obviate the necessity for a search in the computational algorithm. As an important example, the DROLD (least disjoint) family can be used as an effective substitute for the DROLA (least asymmetric) family.

The  $\Lambda$ -equivalent substitution of a constraint-selected family for a search-optimized family enables fast computation of those constraint-selected family members for which the corresponding search-optimized family members would require excessively slow computation. Because of the  $\Lambda$ -equivalence, this substitution can be performed without any loss greater than the tolerance  $\tau(\Lambda)$  for the parameter  $\lambda$  representing the characteristic property of the filter. Sufficiently fast computation of filters within required error tolerances becomes critically important for real-time or on-line adaptive applications.

The spectral factorization approach advocated here for the systematized collection of Daubechies wavelets has been criticized [18,9] for the numerical instabilities associated with finding the roots of a symmetric positive polynomial at high orders. However, the angular parameterization methods, albeit avoiding the root-finding problem, do not guarantee that filters generated by lattices will have other desirable characteristics such as maximal frequency-domain selectivity or minimal time-frequency uncertainty. Although the parameter-space constraint on the angles for  $K = 1$  vanishing moment on the wavelet [9] may insure some time-domain regularity and other desirable characteristics with relevance to low order filters with small  $N$ , it does not necessarily for high order filters with large  $N$ . Searching a parameter space for the corresponding large  $K$  becomes increasingly computationally expensive. Thus, finding

a filter with desirable characteristics becomes more difficult because of the unrestricted search space. Although the angular parameterization of Zou and Tewfik [18] does impose constraints for more than one vanishing moment, they did not present any filter examples for  $K > 2$ .

In contrast, Daubechies wavelets with a wide variety and combination of desirable filter characteristics can be readily computed via spectral factorization as demonstrated in the systematized collection developed in [11,15,17] and reviewed here. Thus, despite the criticism of other authors [18,9] regarding the numerical instabilities inherent in spectral factorization, so far the method remains more useful in generating higher order wavelets with more than one vanishing moment. Clearly, each of the different approaches has advantages and disadvantages. Therefore, the most prudent and practical position to adopt would be that of verifying for each algorithm its utility in terms of the class of filters and range of filter lengths  $N$  for which the algorithm is valid, the possible combinations of desired filter characteristics for which a search can be done, and the computational complexity of the search for filters with those characteristics. As reviewed here, this task has been completed for the Daubechies wavelets computed via spectral factorization.

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Disjoint Sets of Daubechies Polynomial Roots

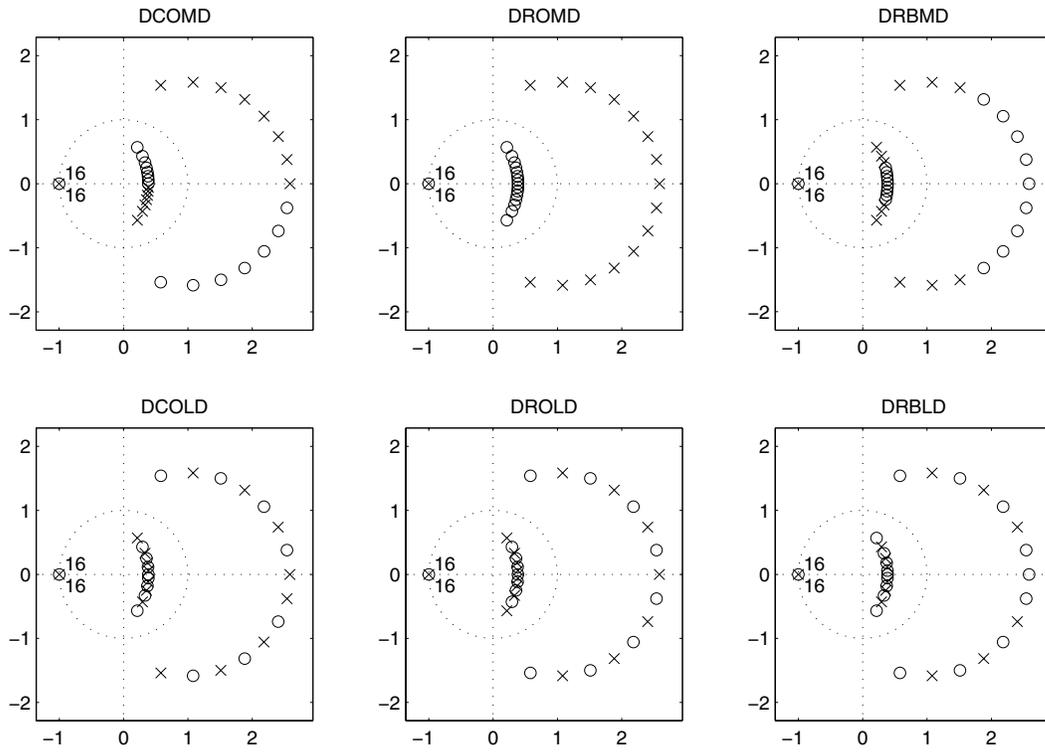


Fig. 1. Examples of disjoint sets of Daubechies polynomial roots.

Wavelets for Disjoint Root Set Examples

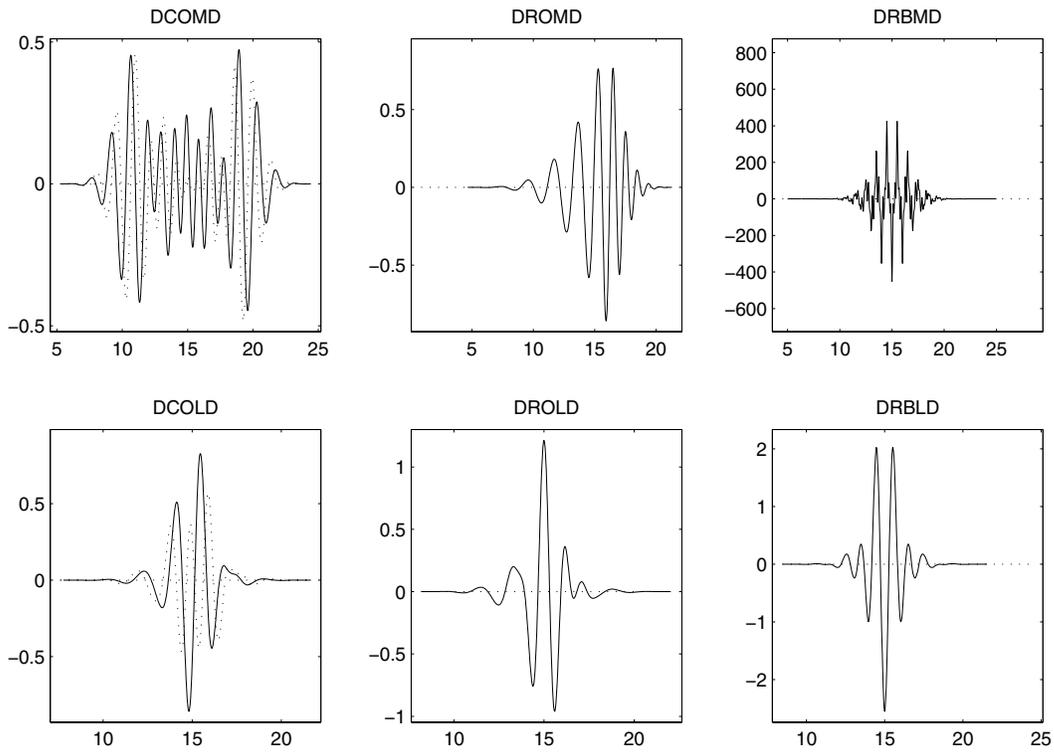


Fig. 2. Analysis wavelets for disjoint root set examples.

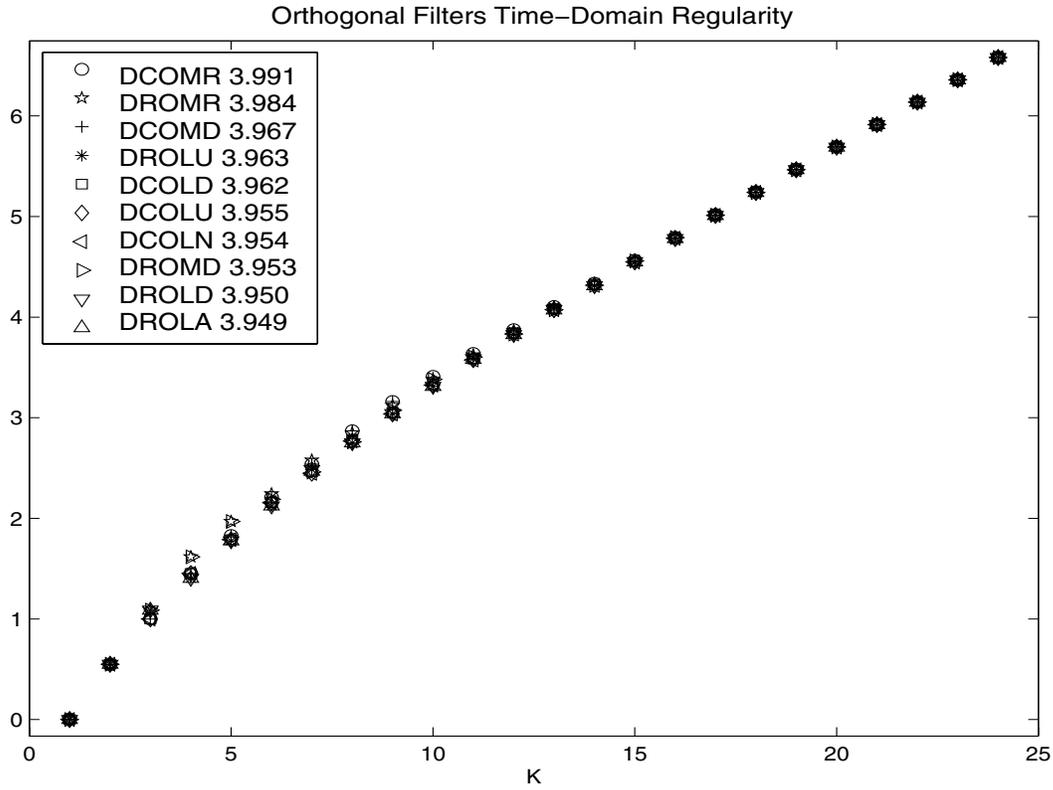


Fig. 3. Time-domain regularity for orthogonal filters.

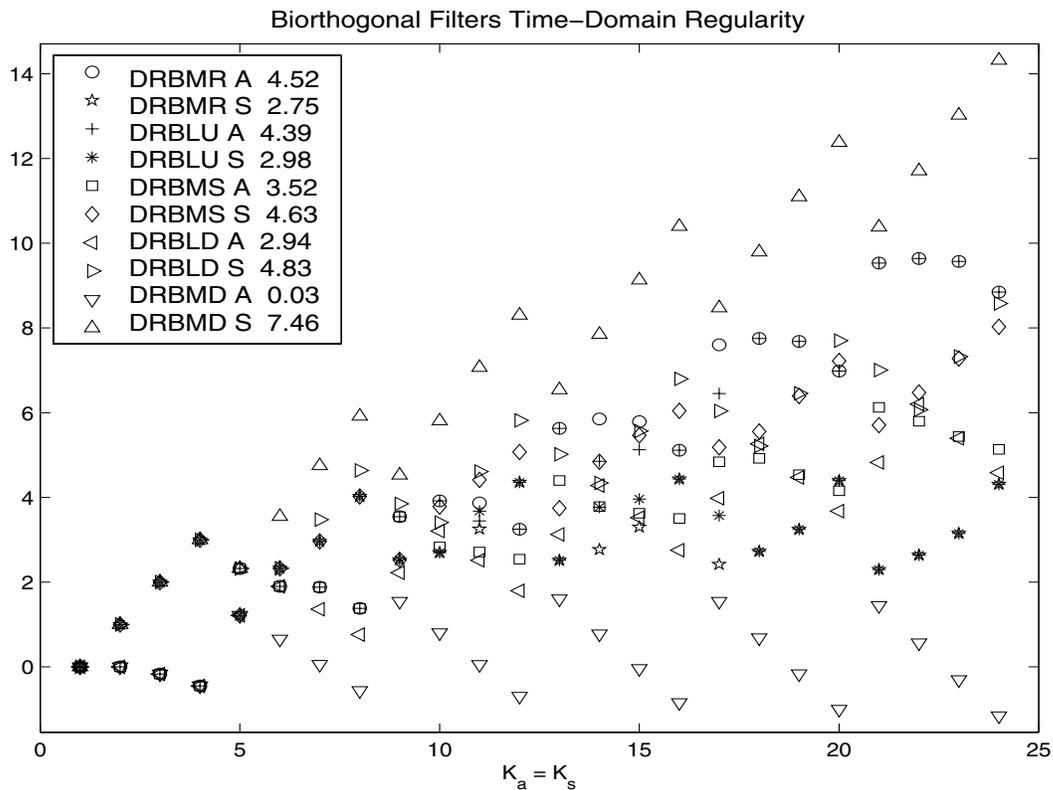


Fig. 4. Time-domain regularity for biorthogonal filters.

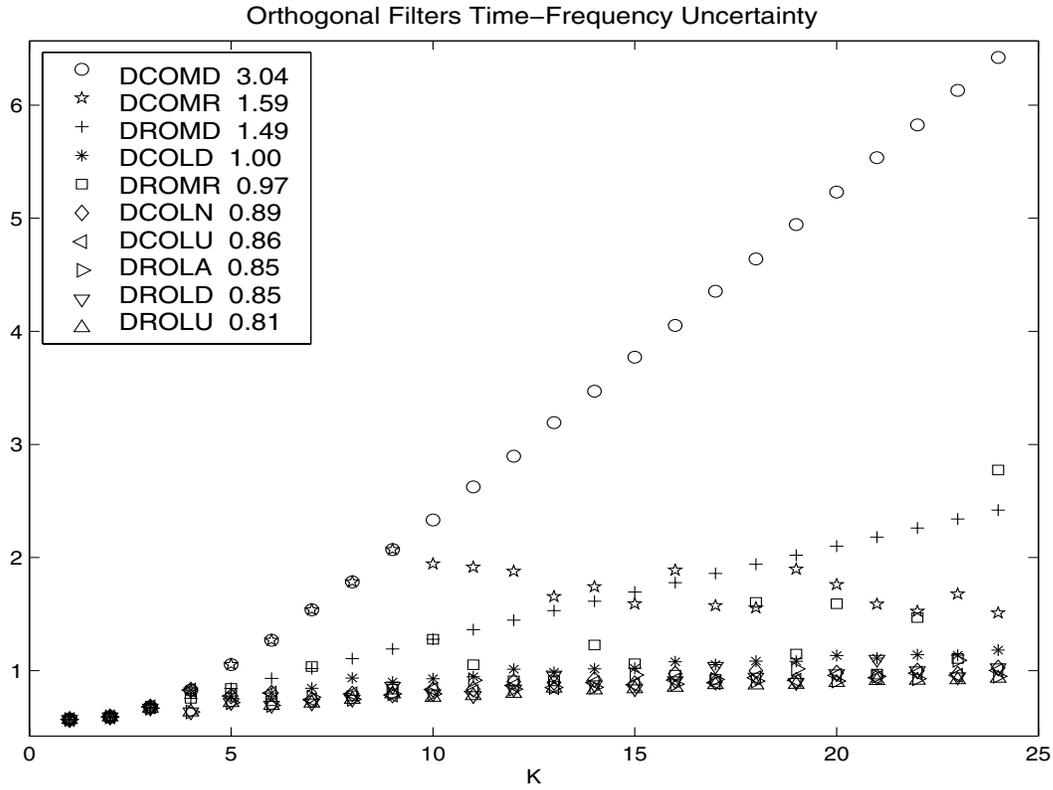


Fig. 5. Time-frequency uncertainty for orthogonal filters.

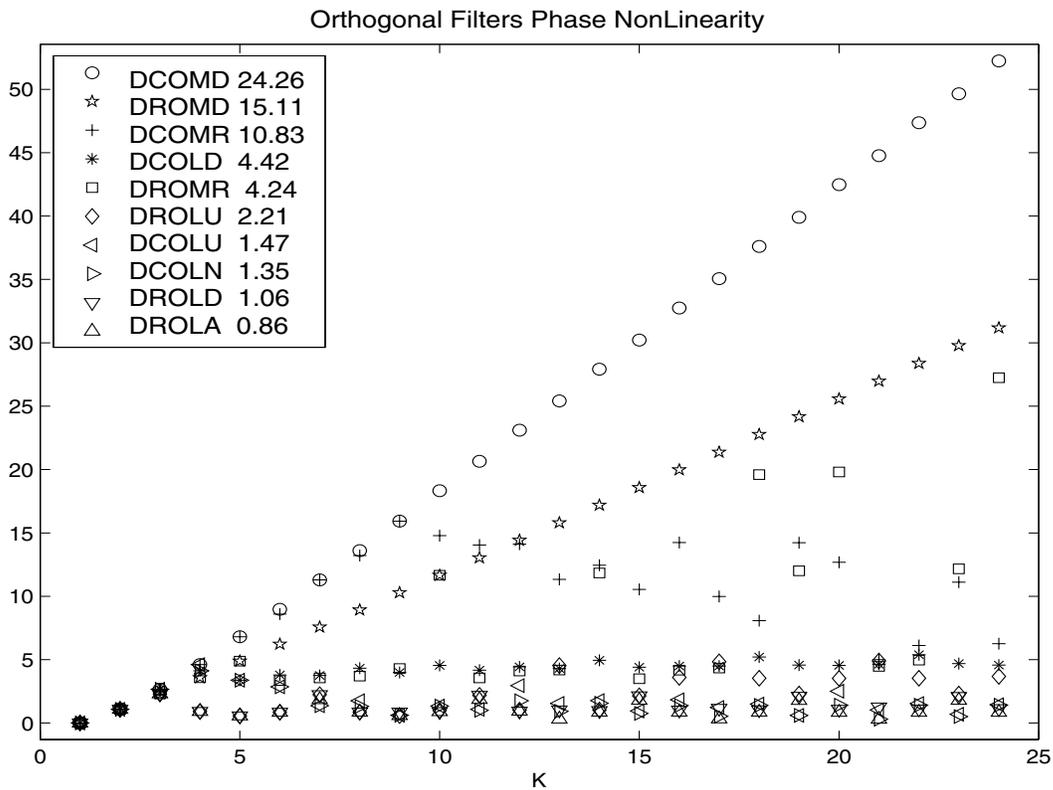


Fig. 6. Phase nonlinearity for orthogonal filters.