A Spectral-Factorization Combinatorial-Search Algorithm Unifying the Systematized Collection of Daubechies Wavelets

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Abstract

A spectral-factorization combinatorial-search algorithm has been developed for unifying a systematized collection of Daubechies minimum length maximum flatness wavelet filters optimized for a diverse assortment of criteria. This systematized collection comprises real and complex orthogonal and biorthogonal wavelets in families within which the filters are indexed by the number of roots at $z = -1$. The main algorithm incorporates spectral factorization of the Daubechies polynomial with a combinatorial search of spectral factor root sets indexed by binary codes. The selected spectral factors are found by optimizing the desired criterion characterizing either the filter roots or coefficients. Daubechies wavelet filter families have been systematized to include those optimized for time domain regularity, frequency domain selectivity, time frequency uncertainty, and phase nonlinearity. The latter criterion permits construction of the orthogonal least and most asymmetric real and least and most symmetric complex filters. Biorthogonal symmetric spline and balanced length filters with linear phase are also computable by these methods.

1 Introduction

Since the discovery of compact orthogonal and biorthogonal wavelets by Daubechies [1], various discussions of the general theory and specific parameterizations of her wavelets have also been published (*cf.* [6] for a literature review). These compact Daubechies wavelets, which have the maximal number of vanishing moments for their finite length, can be implemented as discrete filters that are iterated or auto-convolved to generate approximations of the continuous functions.

A systematic treatment collecting and evaluating all of the Daubechies real and complex orthogonal and biorthogonal wavelets constructed with a single unifying computational algorithm has not yet appeared in the literature. Such an effort was begun by Taswell [5] focusing on wavelets with varying degrees of asymmetry or symmetry that can be derived by spectral factorization of the Daubechies polynomial. Significant advantages of the spectral factorization approach include its generalizability to many different classes and types of wavelets, its suitability for easily interpretable visual displays, and thus its practicality in pedagogy.

Compact asymmetric and symmetric wavelets include the original orthogonal "extremal phase" and "least asymmetric" families as well as the biorthogonal "spline" and "spline variations" families described by Daubechies [1]. As introduced by Taswell [5], these families can be extended and systematized with a generalized flexible yet automated algorithm that permits consistent selection of alternative choices and the identification of filters with optimized parameters. As further developed in this report, these parameters now include phase nonlinearity, time domain regularity, frequency domain selectivity, and time frequency uncertainty, but can be readily extended to include other parameters as optimization criteria. In all cases, a combinatorial search algorithm incorporating a binomial subset selection [4, 5] is used to choose the spectral factors satisfying the required objective criterion defined for each family.

2 Methods

Methods presented here focus on the spectral-factorization combinatorial-search algorithm which requires a separate algorithm for computing the roots of the product filter $\mathcal{P}_{\mathcal{D}}(z)$ or its related form the quotient filter

$$\mathcal{Q}_{\mathcal{D}}(z) = (z+1)^{-2(\mathcal{D}+1)}\mathcal{P}_{\mathcal{D}}(z) \qquad (1)$$

which is the product filter divided by the roots at $z = -1$. Refer to [6] for details.

2.1 Factorization Rules

When considering spectral factorization, the product filter polynomial $\mathcal{P}_{\mathcal{D}}(z)$ with $N_p = 4\mathcal{D} + 3$ coefficients and $K_p = 2\mathcal{D} + 2$ roots at $z = -1$ is factored into the analysis and synthesis filter polynomials $\mathcal{A}(z)$ and $\mathcal{S}(z)$ with N_a and N_s coefficients, and K_a and K_s roots at $z = -1$, respectively. This factorization yields the constraints $N_p = N_a + N_s - 1$ and $K_p = K_a + K_s$ on the lengths of the three filters and their numbers of roots at $z = -1$. Each family of filters in the collection has been named with an identifying acronym followed by the parameters $(N_a, N_s; K_a, K_s)$ in the biorthogonal cases and by $(N;K)$ in the orthogonal cases which require $N \equiv N_a = N_s$ and $K \equiv K_a = K_s$. See Table 1 for a list summarizing the names and designs of the filter families.

With regard to the respective cases of real biorthogonal, real orthogonal, and complex orthogonal, various additional constraints must be imposed. If K_a, K_s, and $K_p = K_a + K_s$ are the numbers of roots at $z = -1$ for $\mathcal{A}(z)$, $\mathcal{S}(z)$, and $\mathcal{P}(z)$, then the corresponding filters have coefficient lengths

$$N_a = K_a + 4n_a^{\mathrm{cq}} + 2n_a^{\mathrm{rd}} + 1 \quad (2)$$

$$N_s = K_s + 4n_s^{\mathrm{cq}} + 2n_s^{\mathrm{rd}} + 1 \quad (3)$$

$$N_p = 2K_p - 1 \quad (4)$$

where n_a^{cq}, n_s^{cq}, n_a^{rd}, and n_s^{rd} are the numbers of complex quadruplets $\{z, z^{-1}, \bar{z}, \bar{z}^{-1}\}$ and real duplets $\{r, r^{-1}\}$ for $\mathcal{A}(z)$ and $\mathcal{S}(z)$. Both n^{cq} and n^{rd} may be whole or half integer. If half integer, then half a complex quadruplet denotes a complex duplet while half a real duplet denotes a real singlet.

For K_a and K_s necessarily both odd or both even, then K_p is always even and $K = K_p/2$ a whole integer determines $n_p^{\mathrm{cq}} = n_a^{\mathrm{cq}} + n_s^{\mathrm{cq}}$ and $n_p^{\mathrm{rd}} = n_a^{\mathrm{rd}} + n_s^{\mathrm{rd}}$ according to $n_p^{\mathrm{cq}} = \lfloor (K-1)/2 \rfloor$ and $n_p^{\mathrm{rd}} = (K-1) \bmod 2$. If K_a and K_s are given, then K_p and K yield n_p^{cq} and n_p^{rd} split into $\{n_a^{\mathrm{cq}}, n_a^{\mathrm{rd}}\}$ and $\{n_s^{\mathrm{cq}}, n_s^{\mathrm{rd}}\}$ and the roots are factored accordingly. For real coefficients, a root z must be paired with its conjugate $\bar{z}$. For symmetric coefficients, a root z must be paired with its reciprocal z^{-1}. For 2-shift orthogonal coefficients, a root z must be *separated* from its conjugate reciprocal $\bar{z}^{-1}$.

Thus, in the real biorthogonal symmetric case, each complex quadruplet $\{z, \bar{z}, z^{-1}, \bar{z}^{-1}\}$ and real duplet $\{r, r^{-1}\}$ must be assigned in its entirety to either $\mathcal{A}(z)$ or $\mathcal{S}(z)$. In the real orthogonal case, each complex quadruplet is split into two conjugate pairs $\{z, \bar{z}\}$ and $\{z^{-1}, \bar{z}^{-1}\}$, while each real duplet is split into two singlets $\{r\}$ and $\{r^{-1}\}$, with one factor assigned to $\mathcal{A}(z)$ and the other to $\mathcal{S}(z)$. The complex orthogonal case is analogous to the real orthogonal case except the complex quadruplets are split into reciprocal pairs $\{z, z^{-1}\}$ and $\{\bar{z}, \bar{z}^{-1}\}$ instead of conjugate pairs. The complex orthogonal symmetric case requires use of complex quadruplets without real duplets.

All orthogonal cases require $K = K_a = K_s = K_p/2$, $n_a^{\mathrm{cq}} = n_s^{\mathrm{cq}} = n_p^{\mathrm{cq}}/2$, and $n_a^{\mathrm{rd}} = n_s^{\mathrm{rd}} = n_p^{\mathrm{rd}}/2$ with $N = N_a = N_s = 2K$. Note that n_p^{rd} can only equal 0 or 1. Therefore, in biorthogonal cases, either $\{n_a^{\mathrm{rd}} = 0,\ n_s^{\mathrm{rd}} = 1\}$ or $\{n_a^{\mathrm{rd}} = 1,\ n_s^{\mathrm{rd}} = 0\}$. However, in orthogonal cases, either $\{n_a^{\mathrm{rd}} = n_s^{\mathrm{rd}} = 0\}$ or $\{n_a^{\mathrm{rd}} = n_s^{\mathrm{rd}} = 1/2\}$ with 1/2 of a duplet denoting a singlet. For all real orthogonal cases as well as those complex orthogonal cases not involving symmetry criteria, K can be any positive integer. For the complex orthogonal least-asymmetric and most-asymmetric cases, K must be a positive even integer. For the complex orthogonal least-symmetric and most-symmetric cases, K must be a positive odd integer.

For the real biorthogonal symmetric cases, K_a and K_s must be both odd or both even. In the biorthogonal symmetric spline case, all additional roots (other than those at $z = -1$ with assignment determined by K_a and K_s) are assigned to the analysis filter leaving the synthesis filter as the spline filter. All other biorthogonal symmetric cases incorporate a root assignment constraint that balances the lengths of the analysis and synthesis filters such that $N_a \approx N_s$ as much as possible. For $K_a = 2i-1$ and $K_s = 2j-1$ both odd with $i, j \in \{1, 2, 3, \dots\}$, balancing of equal filter lengths is possible. In fact, requiring *both* $K_a = K_s$ *and* $N_a = N_s$ is also possible when $N = N_a = N_s = 2K$ with $K = K_a = K_s$ for $\{K = 1 + 4k \mid k = 1, 2, 3 \dots\}$. However, for $K_a = 2i$ and $K_s = 2j$ both even, equal balancing of filter lengths N_a and N_s is not possible. The additional unbalanced roots are assigned to the analysis filter such that $N_a > N_s$ leaving the synthesis filter as the shorter filter.

2.2 Selection Criteria

Selection criteria to be optimized include the phase nonlinearity $\mathrm{pnl}(\mathcal{A})$, time domain regularity $\mathrm{tdr}(\mathcal{A})$, frequency domain selectivity $\mathrm{fds}(\mathcal{A})$, and time frequency uncertainty $\mathrm{tfu}(\mathcal{A})$ [5, 3]. For all of the biorthogonal symmetric balanced-length families (which excludes the biorthogonal symmetric spline family) and for all of the orthogonal families, the selection criterion is optimized for the analysis filter. However, for the biorthogonal balanced-length balanced-regular family, the selection criterion is optimized for both analysis and synthesis filters by maximizing a balancing measure B defined as

$$B(\mathrm{tdr}(\cdot), \mathcal{A}, \mathcal{S}) = \left| \frac{\mathrm{tdr}(\mathcal{A}) + \mathrm{tdr}(\mathcal{S})}{\mathrm{tdr}(\mathcal{A}) - \mathrm{tdr}(\mathcal{S})} \right| \quad (5)$$

when applied to $\mathrm{tdr}(\cdot)$ for $\mathcal{A}$ and $\mathcal{S}$. Phase nonlinearity $\mathrm{pnl}(\mathcal{A})$ does not apply to real biorthogonal filters with linear phase. However, it does apply to the real and complex orthogonal filters. Minimizing or maximizing $\mathrm{pnl}(\mathcal{A})$ for real filters defines the least and most asymmetric families, respectively. In addition, if the parity of K is ignored, then minimizing or maximizing $\mathrm{pnl}(\mathcal{A})$ for complex filters defines the least and most nonlinear families, respectively. See Table 1 for a summary of the filter designs. Note that the DROLD and DROMA families are computed via different factorization and selection methods but should ideally be equivalent. Also note that the DCOLN family is the union of the even-indexed DCOLA and odd-indexed DCOMS families, while the DCOMN family is the union of the even-indexed DCOMA and odd-indexed DCOLS families. Complete details for the algorithms to compute each of the various selection criteria can be found elsewhere [6, 3].

2.3 Unifying Algorithm

All filter families of the systematized collection are generated by the spectral factorization and optimizing combinatorial search incorporated in the following algorithm:

1. Input the identifying name `FiltName` for the family of filters and the indexing design parameters K_a and K_s.

2. Compute $K_p = K_a + K_s$, $\mathcal{D} = K_p/2 - 1$, and the $n_p^{\rm cq} = \lfloor \mathcal{D}/2 \rfloor$ complex quadruplet and $n_p^{\rm rd} = \mathcal{D} \bmod 2$ real duplet roots of the quotient filter $\mathcal{Q}_{\mathcal{D}}(z)$.

3. Determine the factorization rules and selection criterion that define the family of filters named `FiltName`.

4. Compute the splitting number pairs $\{n_a^{\rm cq}, n_s^{\rm cq}\}$ and $\{n_a^{\rm rd}, n_s^{\rm rd}\}$ from $\{n_p^{\rm cq}, n_p^{\rm rd}\}$ for the `FiltName` filter pair indexed by $\{K_a, K_s\}$.

5. Sort the roots in an order convenient for the class of splitting appropriate to the type of filter. For example, all roots of a complex quadruplet should be adjacent with duplets of the quadruplet subsorted according to conjugates or reciprocals depending on the filter type. Assign binary coded labels 0 and 1 to the first and second duplet of each quadruplet. Analogously assign binary codes to the first and second singlet of the real reciprocal duplet if present.

6. Generate the possible binomial subsets for these binary codes [2] subject to the imposed factorization rules and splitting numbers. For example, in the orthogonal case, there are a total of $n_a^{\rm cq} + n_a^{\rm rd}$ binary selections and $2^{n_a^{\rm cq}+n_a^{\rm rd}-1}$ binomial subsets ignoring complements.

7. For each root subset selected by the binomial subset codes, characterize the corresponding filter by the optimization criterion appropriate for the `FiltName` family. These optimization criteria may be any of the numerically estimated filter parameters computed from the roots or the coefficients [6, 3].

8. Search all root subsets to find the one with the optimal value of the desired criterion.

9. Include the K_a and K_s required roots at $z = -1$ for the selected optimal subset of roots intended for the spectral factor $\mathcal{A}(z)$ and for the complementary subset intended for the synthesis spectral factor $\mathcal{S}(z)$.

10. In the orthogonal case, compare the selected (primary) subset of filter roots and coefficients with its complementary subset to choose the one with minimax group delay over the interval $\omega \in [0, 3]$ as the subset for $\mathcal{A}(z)$.

Full searches of all possible combinatorial subsets should be performed for a sufficient number of K indices for the filter family's members to infer the appropriate pattern of binary codes characterizing the family. Using such a pattern permits successful *partial* rather than *full* combinatorial searches. These partial searches provide significant reduction in computational complexity convenient for larger values of K.

3 Results

All filters of all families were demonstrated to meet or surpass requirements for orthogonality, biorthogonality, and reconstruction when tested [3] in 2-band wavelet filter banks. In general, reconstruction errors ranged from "perfect" at $\mathcal{O}(10^{-16})$ to "near-perfect" at $\mathcal{O}(10^{-8})$ as K ranged from $K = 1$ to $K = 24$ for both orthogonal and biorthogonal classes. All filter families were observed to have the optimal values of their defining selection criterion when compared to the other families. Figures 1 and 2 display examples of results for two of the selection criteria, $\mathrm{pnl}(\mathcal{A})$ and $\mathrm{tfu}(\mathcal{A})$, as a function of K for each filter family. Note that the lists in the figure legends order the filter families according to the median values observed for $1 \le K \le 24$. Refer to [6] for a complete catalogue of all results for all of the filter families with both numerical tables of parameter estimates and graphical displays of the filters in the time, frequency, and Z domains.

4 Discussion

An algorithm has been developed to unify all of the diverse families of real and complex orthogonal and biorthogonal Daubechies wavelets. This automated algorithm is valid for any order K of wavelet and insures that the same consistent choice of roots is always made in the computation of the filter coefficients. It is also sufficiently flexible and extensible that it can be generalized to select roots for filters optimized by criteria other than those mentioned here in this report. Systematizing a collection of filters with a mechanism both for generating and evaluating the filters enables the development of filter catalogues with tables of numerical parameter estimates characterizing their properties. Use of these catalogues as a resource enables the investigator to choose an available filter with the desirable characteristics most appropriate to his research problem or development application.

References

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Table 1: Summary of Filter Designs for Systematized Collection of Daubechies Wavelets

Nonorthogonal $\mathcal{P}(z)$	Description	Construction	Index Constraint	
LRNSI$(N_p; K_p; d)$	Symmetric Interpolating	$\mathcal{P}_{\mathcal{L}}(z)$ coefs	$K_p = 2\mathcal{L};\ \mathcal{L} \geq 1$	
DRNSI$(N_p; K_p; d)$	Symmetric Interpolating	via $\mathcal{Q}_{\mathcal{D}}(z)$ roots	$K_p = 2\mathcal{D}+2;\ \mathcal{D} \geq 0$	

Biorthogonal $\mathcal{A}(z)$, $\mathcal{S}(z)$	Description	Factorization	Constraint	Optimization
DRBSS$(N_a, N_s; K_a, K_s)$	Symmetric Spline	$\{z_k \neq -1\}$ to $\mathcal{A}(z)$	even $(K_a + K_s)$	none
DRBLU$(N_a, N_s; K_a, K_s)$	Least Uncertain	conj recip quads	even $(K_a + K_s)$	$\min \mathrm{tfu}(\mathcal{A})$
DRBMS$(N_a, N_s; K_a, K_s)$	Most Selective	conj recip quads	even $(K_a + K_s)$	$\max \mathrm{fds}(\mathcal{A})$
DRBMR$(N_a, N_s; K_a, K_s)$	Most Regular	conj recip quads	even $(K_a + K_s)$	$\max \mathrm{tdr}(\mathcal{A})$
DRBBR$(N_a, N_s; K_a, K_s)$	Balanced Regular	conj recip quads	even $(K_a + K_s)$	$\max B(\mathrm{tdr}(\cdot), \mathcal{A}, \mathcal{S})$

Orthogonal $\mathcal{A}(z)$	Description	Factorization	Constraint	Optimization
DROLD$(N; K)$	Least Delayed	$\{\lvert z_k\rvert < 1\}$ to $\mathcal{A}(z)$	$K \geq 1$	none
DROLU$(N; K)$	Least Uncertain	quads $\rightarrow$ conj dups	$K \geq 1$	$\min \mathrm{tfu}(\mathcal{A})$
DROMR$(N; K)$	Most Regular	quads $\rightarrow$ conj dups	$K \geq 1$	$\max \mathrm{tdr}(\mathcal{A})$
DROLA$(N; K)$	Least Asymmetric	quads $\rightarrow$ conj dups	$K \geq 1$	$\min \mathrm{pnl}(\mathcal{A})$
DROMA$(N; K)$	Most Asymmetric	quads $\rightarrow$ conj dups	$K \geq 1$	$\max \mathrm{pnl}(\mathcal{A})$
DCOLU$(N; K)$	Least Uncertain	quads $\rightarrow$ recip dups	$K \geq 1$	$\min \mathrm{tfu}(\mathcal{A})$
DCOMR$(N; K)$	Most Regular	quads $\rightarrow$ recip dups	$K \geq 1$	$\max \mathrm{tdr}(\mathcal{A})$
DCOLS$(N; K)$	Least Symmetric	quads $\rightarrow$ recip dups	odd $K \geq 3$	$\max \mathrm{pnl}(\mathcal{A})$
DCOMS$(N; K)$	Most Symmetric	quads $\rightarrow$ recip dups	odd $K \geq 3$	$\min \mathrm{pnl}(\mathcal{A})$
DCOLA$(N; K)$	Least Asymmetric	quads $\rightarrow$ recip dups	even $K \geq 4$	$\min \mathrm{pnl}(\mathcal{A})$
DCOMA$(N; K)$	Most Asymmetric	quads $\rightarrow$ recip dups	even $K \geq 4$	$\max \mathrm{pnl}(\mathcal{A})$
DCOLN$(N; K)$	Least Nonlinear	quads $\rightarrow$ recip dups	$K \geq 1$	$\min \mathrm{pnl}(\mathcal{A})$
DCOMN$(N; K)$	Most Nonlinear	quads $\rightarrow$ recip dups	$K \geq 1$	$\max \mathrm{pnl}(\mathcal{A})$

Product filter $\mathcal{P}(z)$ is split into spectral factors for analysis filter $\mathcal{A}(z)$ and synthesis filter $\mathcal{S}(z)$. Names are abbreviated with first character L or D for Lagrange or Daubechies, second character R or C for Real or Complex, and third character N, B, or O for nonorthogonal, biorthogonal, or orthogonal. Filters have length N coefficients and K roots at $z = -1$. Interpolating filters are exact for polynomials of regular degree d. All biorthogonal filters are symmetric. All biorthogonal filters except DRBSS have balanced length. Complex conjugate reciprocal quadruplets are split into conjugate duplets or reciprocal duplets; real reciprocal duplets are split into real singlets but have been omitted from the table; see Sections 2.1 and 2.2 for further explanation of factorization rules and selection criteria.

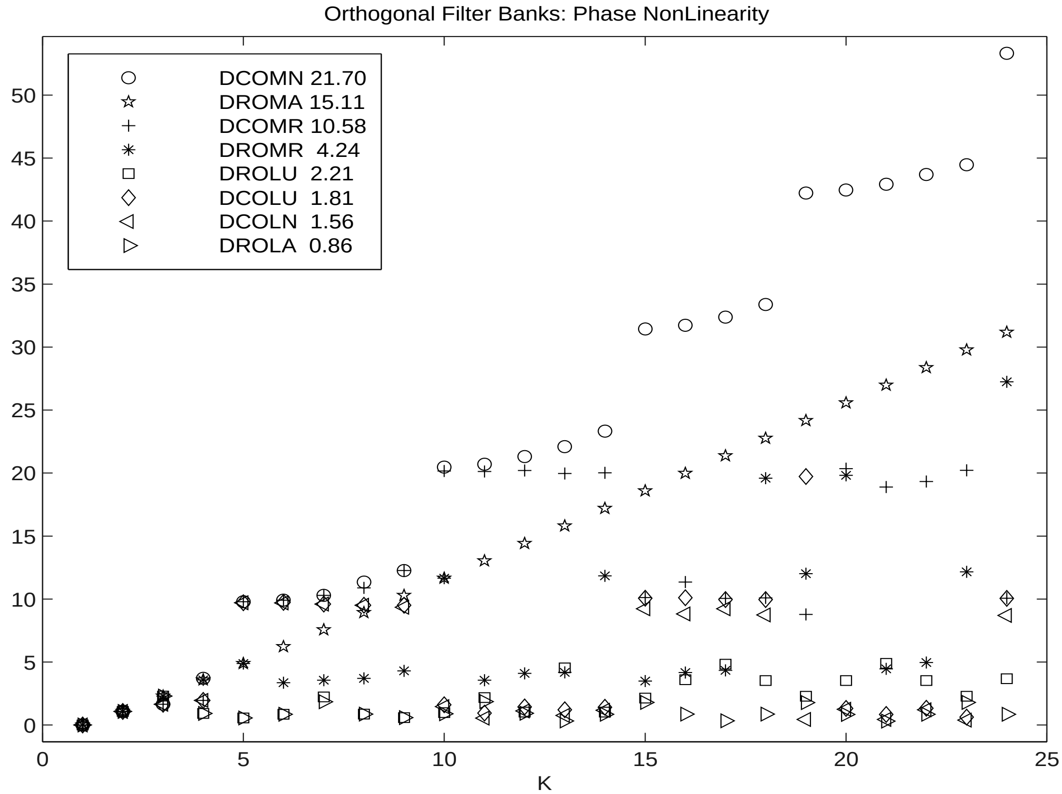

Figure 1: Phase nonlinearity of orthogonal filters.

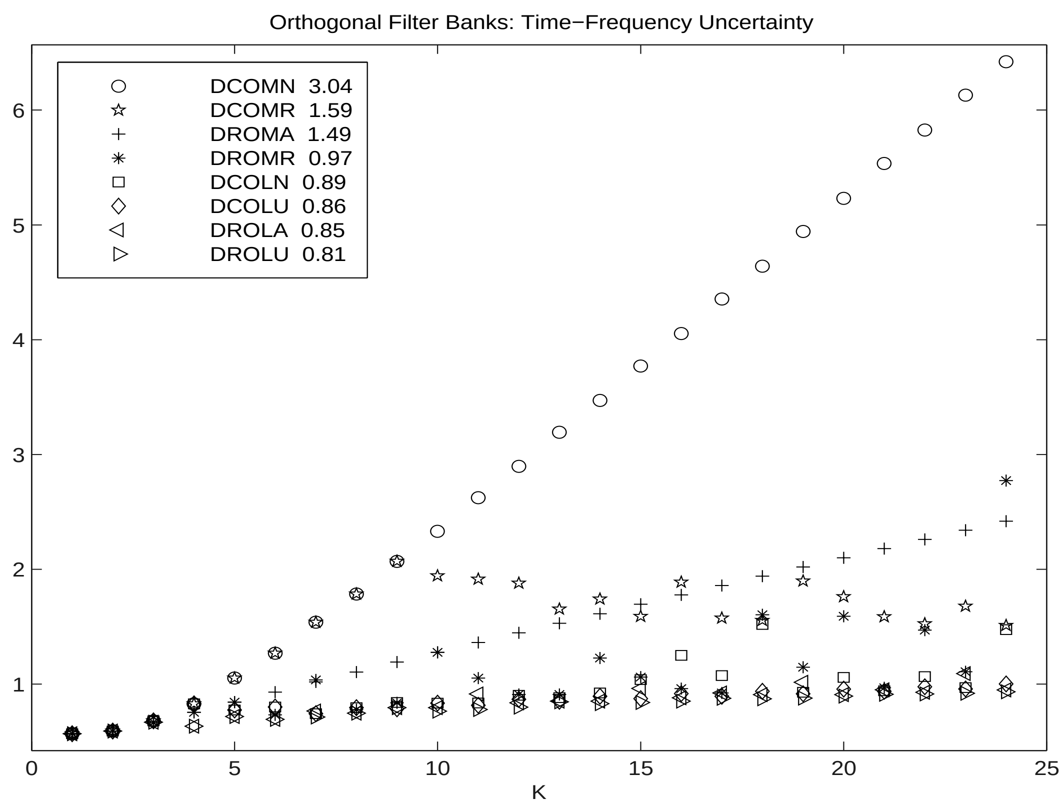

Figure 2: Time frequency uncertainty of orthogonal filters.