Specifications and Standards for Reproducibility of Wavelet Transforms

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Abstract— **As the number of applications and use of wavelet transforms continues to grow, so does the number of classes and variations of wavelet transform algorithms. All of these algorithms incorporate a filter convolution in some implementation, typically, as part of an iterated filter bank. In contrast to implementations of the classical Fourier transform where there is at most a choice of sign and normalization constant in the complex exponential kernel, for wavelet transform algorithms there are multiple choices including both the signs and normalization constants of the wavelet kernels as well as the phase delays or advances of each of the filters in the wavelet filter bank. These algorithmic details, however, are usually not reported in the literature albeit with certain exceptions such as the FBI fingerprint image compression standard. Nevertheless, it is necessary to specify such details in order to insure the reproducibility of results output by each algorithm regardless of its implementation by any programmer working in any language or any engineer designing any DSP chip. This report itemizes a list of choices that must be specified clearly in order to insure the reproducibility of a sequence of transform coefficients generated by a specific wavelet transform algorithm. Moreover, this report proposes a simple but novel solution to the phase alignment problem for wavelet transforms. The general principles of this solution apply in various specific forms to both non-subsampled and critically subsampled wavelet transforms and to both symmetric and asymmetric wavelet filters.**

I. Introduction

Most of the literature on wavelet transforms has discussed the theory of analysis and methods rather than the implementation of algorithms. There have been a few important and notable exceptions such as the papers by Shensa [1] and Rioul and Duhamel [2]. However, these articles discussed algorithmic schemes at a more general level in order to describe them and compare their relative efficiency, rather than algorithmic implementations at a sufficiently detailed level to specify them and insure their reproducibility.

A complete specification of an algorithm may be provided with a detailed pseudo-code template as exemplified in the wavelet transform algorithm published by Taswell and McGill [3] or with sufficiently detailed listing of all mathematical equations and parameters as exemplified by the work of Bradley and Brislawn [4] for the FBI fingerprint image compression standard. However, these published examples remain more the exception than the rule. Systematic development of a standard for the specification and reproducibility of wavelet transform algorithms has not yet been promoted in the wavelet community.

This report presents a proposal for a systematic listing of the principal parameters, choices, and tests that could be specified and performed for filter coefficients, filter convolutions, and wavelet transforms when the investigator wishes to guarantee reproducibility and verifiability regardless of computing platform and programming language. The specification of the filter convolutions, the phase delays and advances of the filters in the filter bank, and the treatment of the ends of the signal remains a central issue relevant to algorithms for finite-length signals. In the introduction to his paper [5], Brislawn provides a comprehensive historical review of the various convolution types available. However, reporting of such details is often neglected. To emphasize the importance of specifying these convolution details, this paper also presents a framework for reporting them and demonstrates the use of this framework with a simple yet novel solution to the phase alignment problem.

II. Methods

Algorithms are specified here by building heirarchical modular components for filter coefficients, filter convolutions, and wavelet transforms in which each stage is detailed with all necessary choices to insure reproducibility and verifiability. The specification outlined here assumes that the wavelet transform can be implemented as an iterated multirate filter bank algorithm. Complete algorithmic details for all of the methods mentioned here will be available in a forthcoming book [6].

A. Filter Coefficients

Consider an M-band analysis/synthesis filter bank system with uniform down/up sampling rate R. This system has M analysis filters with impulse responses $\mathbf{h}_m \equiv h_m(n)$, M downsamplers and upsamplers operating at rate R, and M synthesis filters $\mathbf{g}_m \equiv g_m(n)$ where $m = 0, 1, \ldots, M-1$ is the band index and $n = 0, 1, \ldots, N-1$ is the time index. Here N is the length of the longest filter in the filter banks with $N = LR$ an integer multiple L of the rate R. The first non-zero coefficient of each filter is indexed at time $n = 0$ and any filter shorter than length N is padded with additional zeros. The filter coefficients can then be represented as the matrices $\mathbf{H} = [h_{nm}]$ and $\mathbf{G} = [g_{nm}]$ with time index n increasing down the rows and band index m increasing across the columns. This convention permits columnwise data analysis for the band filters in each of the columns.

A minimal specification of the filter bank coefficients requires either a) actual tabulation of the coefficient matrices $\mathbf{H}$ and $\mathbf{G}$, or b) specific definition of the computational algorithm that generates the coefficient matrices with sufficient detail to clarify choices of signs and normalization constants. Assuming that $\mathbf{H}$ and $\mathbf{G}$ have been unequivocally specified, additional informative characterization of the filter banks may also include a) the accuracy and precision of the numerical coefficients relative to their theoretical values, b) various norms, moments, or other measures of each individual filter in the filter banks, and c) the system delay Δ and reconstruction error ϵ for an impulse processed through the system. A simple modification of the method devised by Nayebi *et al* [7] provides the most convenient and general approach for testing the filter coefficients and computing Δ and ϵ. Finally, the filters may be tested numerically for well-known properties such as orthogonality and energy conservation.

B. *Filter Convolutions*

Under the assumptions validating the noble identities, the order of analysis filters and downsamplers can be exchanged, and similarly the order of upsamplers and synthesis filters can be exchanged [8]. Moreover, for computational efficiency, each pair of operations can be integrated into a single convolution operation called downscaling for the composition of analysis filtering and downsampling, and upscaling for the composition of upsampling and synthesis filtering [9]. Thus, for the purposes of this exposition, the operations will be denoted with the matrices $\mathbf{T}(\mathbf{h}_m)$ for filtering with the m^{th} analysis filter $\mathbf{h}_m$, $\mathbf{D}$ for downsampling, $\mathbf{D}_m \equiv \mathbf{D} \cdot \mathbf{T}(\mathbf{h}_m)$ for downscaling with $\mathbf{h}_m$, $\mathbf{U}$ for upsampling, $\mathbf{T}(\mathbf{g}_m)$ for filtering with the m^{th} synthesis filter $\mathbf{g}_m$, and $\mathbf{U}_m \equiv \mathbf{T}(\mathbf{g}_m) \cdot \mathbf{U}$ for upscaling with $\mathbf{g}_m$. Since the matrices $\mathbf{T}$ are banded Toeplitz matrices implementing standard (linear or circular) convolution, the matrices $\mathbf{D}_m$ and $\mathbf{U}_m$ are block Toeplitz matrices.

Using this matrix notation, the filter convolutions can be implemented and studied as multiplications of the finite-length signal data $\mathbf{X}$ with the finite-size downscaling matrices $\mathbf{D}_m$ to obtain the decomposition bands $\mathbf{Y}_m = \mathbf{D}_m \mathbf{X}$, and then with the upscaling matrices $\mathbf{U}_m$ to obtain the reconstruction bands $\hat{\mathbf{X}}_m = \mathbf{U}_m \mathbf{Y}_m$. Summing these outputs yields the final reconstruction $\hat{\mathbf{X}} = \sum_m \hat{\mathbf{X}}_m$. In this representation, single- and multi-channel data correspond to $\mathbf{X}$ with single and multiple columns, respectively. Discussion of the matrix representation of the filter convolutions suffices to fix issues related to reproducibility. Again, this objective is defined here principally as the requirement that a given sequence of transform coefficients be computed reproducibly for a given sequence of signal coefficients. Thus, issues related to efficiency (such as matrix-filter versus vector-filter implementations [3] and time-domain versus frequency-domain implementations [2]) are not considered here other than as already mentioned at the beginning of this section.

It is, however, the finite size of the filter convolution and data matrices that does directly impact reproducibility of the wavelet transform. This finiteness imposes the necessity to consider the treatment of the ends of the signal, not only with regard to the choice of the type of convolution such as zero-extended [3], circularly-periodized [3], linearly-extended [10], symmetrically-reflected [5], or boundary-adjusted [11], but also with regard to the choices of phase delays and advances for the convolution. To specify the filter convolutions reproducibly, it is thus necessary to clarify unambiguously the convolution type and delays imposed on each filter band in the filter bank.

All of the different kinds and variations of convolutions can be incorporated in the following general framework described here with analysis phase delays α_{im}, synthesis phase delays β_{im}, and several additional matrix operators: the pre-processing or extension matrix $\mathbf{E}$, the shift matrix $\mathbf{S}$, and the post-processing or restriction matrix $\mathbf{R}$. Then the m^{th} analysis downscaling and synthesis upscaling matrices can be redefined as

$$\begin{aligned} \mathbf{D}_m &\equiv \mathbf{R} \cdot \mathbf{D} \cdot \mathbf{T}(\mathbf{h}_m, \alpha_{2m}) \cdot \mathbf{E}(\alpha_{1m}) \\ \mathbf{U}_m &\equiv \mathbf{R} \cdot \mathbf{S}(\beta_{3m}) \cdot \mathbf{T}(\mathbf{g}_m, \beta_{2m}) \cdot \mathbf{U} \cdot \mathbf{E}(\beta_{1m}) \end{aligned}$$

for a scheme intended to impose a perfect reconstruction result $\mathbf{I} = \sum_m \mathbf{U}_m \cdot \mathbf{D}_m$ on a single-level decomposition and reconstruction whenever possible. Note that $\mathbf{S}(\beta_{3m})$ is a final shift necessary to account for the shifts resulting from the other operators and system delay Δ. This scheme assumes zero delays on the $\mathbf{D}$, $\mathbf{U}$, and $\mathbf{R}$ operators.

Phase alignment of peaks of polyphase components of bands in the transform domain relative to the signal domain can be accomplished by the simple introduction of two more shift operators and delay parameters in the scheme

$$\begin{aligned}\mathbf{D}_m &\equiv \mathbf{S}(\alpha_{3m})\mathbf{RDT}(\mathbf{h}_m, \alpha_{2m})\mathbf{E}(\alpha_{1m})\\ \mathbf{U}_m &\equiv \mathbf{RS}(\beta_{4m})\mathbf{T}(\mathbf{g}_m, \beta_{3m})\mathbf{UE}(\beta_{2m})\mathbf{S}(\beta_{1m})\end{aligned}$$

which ideally should require that the final downscaling rotation (circular shift) $\mathbf{S}(\alpha_{3m})$ and the initial upscaling inverse rotation $\mathbf{S}(\beta_{1m})$ yield the identity $\mathbf{I} = \mathbf{S}(\beta_{1m}) \cdot \mathbf{S}(\alpha_{3m})$. Thus, imposing $\beta_{1m} = -\alpha_{3m}$ eliminates one of the additional parameters, and absorbing $\mathbf{S}(\beta_{1m})$ into $\mathbf{E}(\beta_{2m})$ eliminates one of the additional operators. Relabeling indices yields the scheme

$$\begin{aligned}\mathbf{D}_m &\equiv \mathbf{S}(\alpha_{2m})\mathbf{RDT}(\mathbf{h}_m, \alpha_{1m})\mathbf{E}(\alpha_{3m})\\ \mathbf{U}_m &\equiv \mathbf{RS}(\beta_{3m})\mathbf{T}(\mathbf{g}_m, \beta_{1m})\mathbf{UE}(\beta_{2m})\end{aligned}$$

as a general framework sufficient to account for the various convolution types. This particular indexing convention was adopted for WavBox 4.4 Software [6] used to produce the results reported in Section III. Detailed algorithms including pseudo-code templates for these phase aligned convolution types will be available elsewhere [12], [6].

Thus, a minimal specification for reproducibility of the filter convolutions requires a) the convolution type including the composition sequence of the various operators as in the general framework above, b) the algorithms for generating the operators with particular attention to the extension operator $\mathbf{E}$ for a given extension or boundary treatment type, c) any auxiliary parameters or boundary filters necessary for $\mathbf{E}$, d) restriction length parameters necessary for the restriction operator $\mathbf{R}$, and e) the phase delays necessary for any of the $\mathbf{E}$, $\mathbf{T}$, and $\mathbf{S}$ operators used by the convolution type (or the algorithms for setting the phase delays). Additional characterization for verifiability of the filter convolutions may also include: a) comparison of results with known sequences of transform coefficients for given sequences of test signal coefficients, b) the reconstruction error ε for the test signals resulting from use of the filter convolutions as a single-level decomposition and reconstruction, and c) various other measures designed to reveal properties of the convolution type.

A simple test signal called a triple M-spike has been designed to enable visualization of various aspects of the convolution including the behavior of the polyphase components in response to the filters and the boundary treatment. This signal has M-channels in which each channel has an impulse near the beginning, middle, and end of the signal, but the impulses for each channel are shifted relative to each other by one time index. Thus, each channel is intended to test a different polyphase component. In conjunction with this test signal, a measure called the energy shift ratio ρ has been defined to track the energy displaced by the phase shifts of the convolution. This measure has values in the range $0 \leq \rho \leq 1$ with a value of $\rho = 0$ indicating that no energy has been displaced.

C. Wavelet Transforms

Given analysis and synthesis filter bank coefficients specified by $\mathbf{H}$ and $\mathbf{G}$ (Section II-A) used to construct downscaling and upscaling filter convolution operators specified by $\mathbf{D}_m$ and $\mathbf{U}_m$ (Section II-B), then a wavelet transform algorithm can be specified as the procedure by which $\mathbf{D}_m$ and $\mathbf{U}_m$ are used iteratively to process the signal to generate the transform. For an M-band wavelet transform which iterates on the low-pass band indexed $m = 0$, a pseudo-code template for the forward transform algorithm can be written as

$$\begin{array}{l}\mathbf{Y}_0^0 = \mathbf{X}\\ \text{for } l = 0 : L-1\\ \quad \text{for } m = 0 : M-1\\ \quad\quad \mathbf{Y}_m^{l+1} = \mathbf{D}_m^l \mathbf{Y}_0^l\\ \quad \text{end}\\ \text{end}\end{array}$$

and for the inverse wavelet transform algorithm as

$$\begin{array}{l}\hat{\mathbf{X}}_0^L = \mathbf{Y}_0^L\\ \text{for } l = L : -1 : 1\\ \quad \hat{\mathbf{X}}_0^{l-1} = \mathbf{U}_0^l \hat{\mathbf{X}}_0^l + \sum_1^{M-1} \mathbf{U}_m^l \mathbf{Y}_m^l\\ \text{end}\\ \hat{\mathbf{X}} = \hat{\mathbf{X}}_0^0\end{array}$$

with specific algorithms requiring definition of the data structures used for storage of the coefficients (or alternatively, the sequence of coefficients in an output file) in a manner analogous to the example published in ACM TOMS Algorithm 735 [3].

Thus, a minimal specification for reproducibility of a wavelet transform requires a) filter bank coefficients $\mathbf{H}$ and $\mathbf{G}$, b) filter convolution operators $\mathbf{D}_m^l$ and $\mathbf{U}_m^l$, c) algorithmic scheme by which convolution operators are iterated, d) parameter L for number of levels of iteration, and e) transform coefficient data structures and location of coefficients within the data structures or file output sequences. Additional characterization for verifiability of the wavelet transform may also include: a) known sequences of transform coefficients for given sequences of test signal coefficients, and b) the reconstruction error $\mathcal{E}$ for the test signals under various norms and conditions. For example, degradation of the signal can be tracked through multiple cycles of decomposition and reconstruction:

$$
\begin{aligned}
&\hat{\mathbf{X}} = \mathbf{X} \\
&\text{for } k = 1:K \\
&\quad \mathbf{Y} = \mathrm{fwt}(\hat{\mathbf{X}}) \\
&\quad \hat{\mathbf{X}} = \mathrm{iwt}(\mathbf{Y}) \\
&\quad \mathcal{E}(k) = \mathrm{wtre}(\mathbf{X}, \hat{\mathbf{X}}) \\
&\text{end}
\end{aligned}
$$

where the function *fwt* is the forward wavelet transform, *iwt* the inverse wavelet transform, and *wtre* the wavelet transform reconstruction error. Plots of $\mathcal{E}(k)$ versus k can be used to obtain empirical estimates of the error growth rates.

III. Results

Numerical and graphical results reported here were computed with WavBox 4.4a Software [6] for a wavelet filter bank system with $M = R = 2$.

A. Filter Coefficients

The asymmetric orthogonal Daubechies' filters of length $N = 12$ with 4 vanishing moments on both scaling and wavelet filters [13, page 261] were normalized in the ℓ^2-norm to one with the sign of h_{00} taken as negative. Values of $\Delta = 11$ and $\epsilon = 9.487 \times 10^{-12}$ were obtained with the modified Nayebi-Barnwell-Smith perfect reconstruction test [7].

B. Filter Convolutions

A circularly-periodized convolution type was chosen for the single-level decomposition and reconstruction steps and was tested with a triple M-spike signal. Figure 1 displays two different phase variants of this convolution type: a causal analysis variant called peak non-aligned with phase delays $\alpha = [0,0;0,0;0,0]$ and $\beta = [0,0;0,0;-11,-11]$, and an anti-causal analysis variant called peak near-aligned with phase delays $\alpha = [1,1;-4,-3;0,0]$ and $\beta = [0,0;4,3;-12,-12]$. The reconstruction errors for both phase variants were $\varepsilon = 9.147 \times 10^{-12}$. The energy shift ratios were $\rho = 9.989 \times 10^{-1}$ and $\rho = 4.674 \times 10^{-2}$ for the non- and near-aligned variants, respectively.

C. Wavelet Transforms

The single level steps were iterated to $L = 4$ levels, and tested for $K = 100$ cycles of forward and inverse transforms. Figure 2 displays log-log plots of the reconstruction error $\mathcal{E}(k)$ as a function of k. Linear regression estimates of the slopes of the error curves for each of the ℓ^1, ℓ^2, and ℓ^∞ error norms resulted in values of 1.000 thus yielding the empirical observation $\log_{10}\mathcal{E}(k) = \mathcal{E}(1) + \log_{10} k$. Values of $\mathcal{E}^1(1) = 5.001 \times 10^{-9}$, $\mathcal{E}^2(1) = 1.968 \times 10^{-10}$, and $\mathcal{E}^\infty(1) = 2.721 \times 10^{-11}$ were obtained for the ℓ^1, ℓ^2, and ℓ^∞ error norms, respectively.

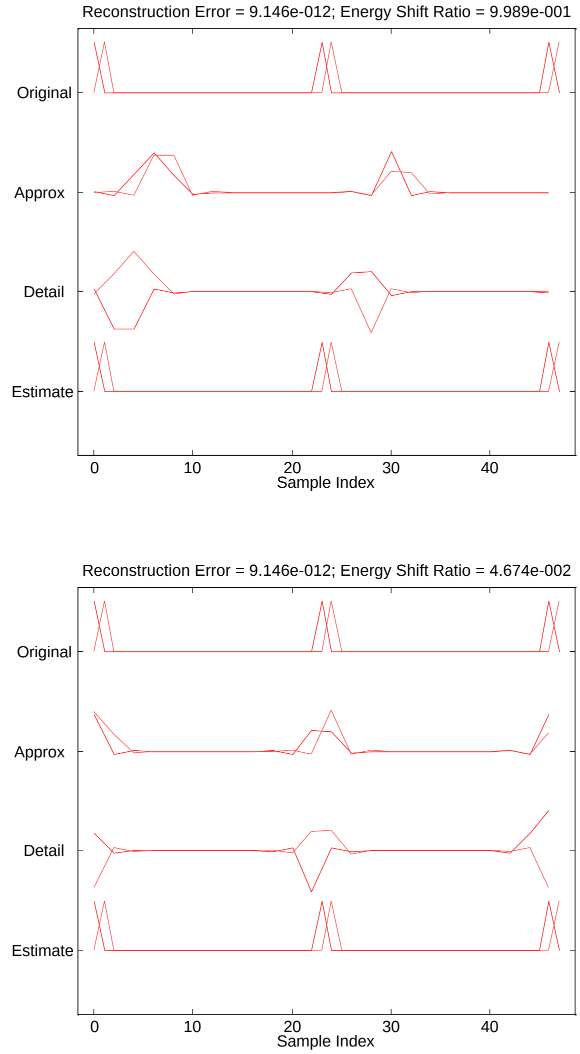

Fig. 1. Single-level decomposition and reconstruction with circularly-periodized convolution for triple M-spike M-channel test signal with $M = 2$. Top: peak non-aligned phase; Bottom: peak near-aligned phase.

IV. Discussion

A proposal for a standard for specifying wavelet transforms cannot be limited to reporting the error from tests of perfect reconstruction after transforming and inverse transforming. While these tests are necessary, they are not sufficient and do not verify that the correct sequence of transform coefficients has been generated. A similar argument applies to tests of energy conservation for energy conserving transforms because transform coefficient sequences with

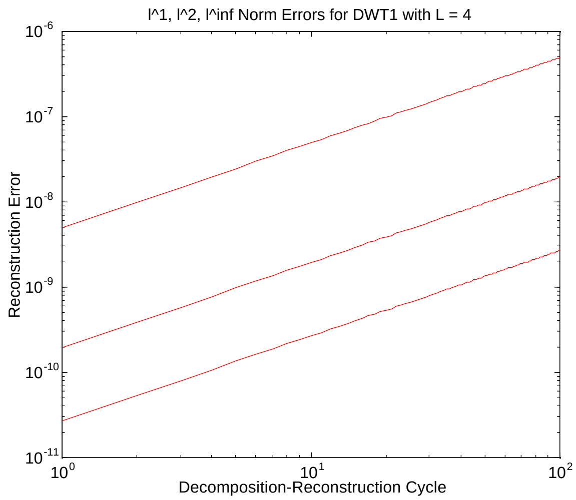

Fig. 2. Plots of $\log_{10} \mathcal{E}(k)$ versus $\log_{10} k$ for ℓ^1, ℓ^2, and ℓ^∞ error norms for a wavelet transform with $L = 4$.

different signs and phases may have the same energy. Thus, there is only one way to insure reproducibility of a wavelet transform algorithm and verify its correct implementation: 1) specify it completely with a sufficiently detailed combination of mathematical equations, choices of parameters, and pseudo-code templates for the algorithm, 2) verify it by comparing results with known sequences of output transform coefficients for given input signal data, and then 3) verify the inverse transform algorithm with tests for perfect reconstruction.

The elements of the standard should specify enough information to enable the algorithm to be implemented and to yield results reproducibly in a consistent manner independently of computing platform and programming language. In this paper, elements of both minimal specifications and additional characterizations were listed for three heirarchical stages of the algorithms: 1) filter coefficients, 2) single-level filter convolutions, and 3) and iterated multi-level wavelet transforms. The *minimal specifications* are required for reproducibility of results whereas the *additional characterizations*, although informative and useful, are not. Whether an element of the standard is declared to be in the former or the latter category can be debated. It will depend on whether the element is considered to be a constituent of the specification or the verification. For example, error tolerance limits could be established as a requirement of the specification, or errors could be merely reported as a characterization of the verification.

The standard should also provide a general framework for specifying the many different types of convolution and their phase delay variants. Different phase delay variants, such as the near-aligned solution introduced here, can affect the results of methods based on wavelet transforms [12], and thus should be reported. Not every convolution type requires all of the operators and delays of the scheme proposed here. If the general framework is nevertheless retained for the specification of all cases, those operators not needed for a particular convolution type can simply be set to identity matrices with zero delays. Of course, a computationally efficient implementation would eliminate this redundancy. However, the general framework proposed here is designed to specify and test reproducibility rather than efficiency.

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