Computational Algorithms for Daubechies Least-Asymmetric, Symmetric, and Most-Symmetric Wavelets

Carl Taswell (ctaswell@ucsd.edu)
Laboratory for Functional Neuroimaging
UCSD School of Medicine, La Jolla, CA 92093-0603

Abstract

Computational algorithms have been developed for generating min-length max-flat FIR filter coefficients for orthogonal and biorthogonal wavelets with varying degrees of asymmetry or symmetry. These algorithms are based on spectral factorization of the Daubechies polynomial with a combinatorial search of root sets selected by a desired optimization criterion. Daubechies filter families were systematized to include Real Orthogonal Least Asymmetric (DROLA), Real Biorthogonal symmetric balanced Most Regular (DRBMR), Complex Orthogonal Least Asymmetric (DCOLA), and Complex Orthogonal Most Symmetric (DCOMS). Total phase nonlinearity was the criterion minimized to select the roots for the DROLA, DCOLA, and DCOMS filters. Time-domain regularity was used to select the roots for the DRBMR filters (which have linear phase only). New filters with distinguishing features are demonstrated with examples.

1 Introduction

Compact maximally flat wavelets with varying degrees of asymmetry or symmetry can be derived by spectral factorization of the Daubechies polynomial. These wavelets, which have the maximal number N of vanishing moments for their finite length, include the original orthogonal "extremal-phase" and "least-asymmetric" families as well as the biorthogonal "spline-variations" family described by Daubechies [1, 3, 2]. In this report, these families are extended and systematized with automated algorithms that permit consistent selection of alternative choices and the identification of filters with optimal criteria.

Daubechies [3] defined the total nonlinear phase as the criterion she used to select the spectral factors for the least-asymmetric family in the case of real orthogonal wavelets. The definition of total nonlinear phase is extended here with a criterion used to select the spectral factors for the least-asymmetric and most-symmetric families in the case of complex orthogonal wavelets. However, nonlinear phase does not exist and cannot be used for the symmetric family in the case of real biorthogonal wavelets. Therefore, the combined criteria of maximally balanced filter length and regularity for both analysis and synthesis wavelets are used to select the spectral factors for this symmetric family.

In all four cases (real orthogonal least-asymmetric, real biorthogonal symmetric, complex orthogonal least-asymmetric, and complex orthogonal most-symmetric), a combinatorial search algorithm incorporating a binomial subset selection [8] is used to choose the spectral factors meeting the required criteria defined for each family. Computational algorithms for the generation of the filter coefficients and evaluation of their numerical properties are presented together with representative examples chosen for their distinguishing characteristics.

2 Methods

Wavelet filter coefficients were generated by a revision of the combination of methods described previously [8]. These methods incorporate in sequence the following steps: a) construct the Daubechies polynomials [2, pg. 171, eq. 6.1.12], b) compute their zeros [6], c) include the required zeros at $z = -1$ for the Product filter $P(z)$, d) factor $P(z)$ to alternative Analysis and Synthesis filters $A(z)$ and $S(z)$, e) label the roots of these factors with a binary code and generate the possible combinatorial subsets for these binary codes, f) characterize each root subset and its corresponding filter by the total phase nonlinearity or another desired property such as the time-domain regularity, g) search the root subsets to find the one with the optimal value of the desired criterion, h) compare the selected (primary) subset with its complementary subset to choose the one with minmax group delay as the root subset for $A(z)$ (applicable only in orthogonal cases), and i) compute the wavelet coefficients from the scalet coefficients obtained from each of $A(z)$ and $S(z)$.

In particular, if N_a, N_s, and $N_p = N_a + N_s$ are the numbers of zeros at $z = -1$ for $A(z)$, $S(z)$, and $P(z)$, respectively, then the corresponding filters have coefficient lengths $L_a = N_a + 4C_a + 2R_a + 1$, $L_s = N_s + 4C_s + 2R_s + 1$, and $L_p = 2N_p - 1$ where C_a, C_s, R_a, and R_s are the numbers of complex quadruplets and real duplets for $A(z)$ and $S(z)$. Moreover, for N_a and N_s necessarily both odd or both even, then N_p is always even and $N = N_p/2$ a whole integer determines $C_p = C_a + C_s$ and $R_p = R_a + R_s$ according to $C_p = \text{floor}((N-1)/2)$ and $R_p = \text{rem}((N-1)/2)$. Thus, if N_a and N_s are given, then N_p and N yield C_p and R_p split into $\{C_a, R_a\}$ and $\{C_s, R_s\}$ and the roots are factored accordingly. For real coefficients, both a root z and its conjugate $\bar{z}$ must be paired. For linear phase symmetric coefficients, both a root z and its reciprocal z^{-1} must be paired.

Thus, in the biorthogonal symmetric case, each complex quadruplet $\{z, \bar{z}, z^{-1}, \bar{z}^{-1}\}$ and real duplet $\{z, z^{-1} \mid z = x + iy,\ y = 0\}$ must be assigned in its entirety to either $A(z)$ or $S(z)$. In the real orthogonal case, each complex quadruplet is factored into two conjugate pairs $\{z, \bar{z}\}$ and $\{z^{-1}, \bar{z}^{-1}\}$, while each real duplet is factored into two singlets $\{z\}$ and $\{z^{-1}\}$, with one factor assigned to $A(z)$ and the other to $S(z)$. The complex orthogonal case is analogous to the real orthogonal case except the complex quadruplets are factored into reciprocal pairs $\{z, z^{-1}\}$ and $\{\bar{z}, \bar{z}^{-1}\}$ instead of conjugate pairs. The orthogonal cases require $N = N_a = N_s = N_p/2$, $C_a = C_s = C_p/2$ and $R_a = R_s = R_p/2$ with $L = L_a = L_s = 2N$. (Note that R_p can only equal 0 or 1. Therefore, in the biorthogonal case, either $\{R_a = 0,\ R_s = 1\}$ or $\{R_a = 1,\ R_s = 0\}$. However, in the orthogonal case, either $\{R_a = R_s = 0\}$ or $\{R_a = R_s = 1/2\}$ with 1/2 of a duplet denoting a singlet.) For the real orthogonal least-asymmetric case, N can be any positive integer, whereas complex orthogonal least-asymmetric and most-symmetric require positive N even and odd, respectively.

With regard to the nonlinear phase contribution for the complex conjugate pair $\{z, \bar{z}\}$, Daubechies [2, pg. 255] provided a derivation for the formula

$$\arctan\left[\frac{(r^2-1)\sin(\omega)}{(r^2+1)\cos(\omega) - 2r\cos(\alpha)}\right]$$

where her notation has been modified with use of $z = re^{i\alpha}$. This method can be extended with an analogous derivation for the complex reciprocal pair $\{z, z^{-1}\}$ resulting in the formula

$$\arctan\left[\frac{(r-r^{-1})\sin(\alpha)}{(r+r^{-1})\cos(\alpha) - 2\cos(\omega)}\right].$$

Total Phase NonLinearity, denoted $\text{pnl}(H)$ for the filter $H(z)$, was computed as the discrete approximation of the $L^1[0, 2\pi]$ integral of the sum of the contributions from the roots for $H(z)$.

Time-Domain Regularity $\text{tdr}(H)$ was computed by the method of Rioul [5]. Discrete Time-Frequency Uncertainty $\text{tfu}(H)$ was computed by the method of Haddad *et al* [4] and reported as the area of the Heisenberg uncertainty box with width $2\sigma_n$ and height $2\sigma_\omega$ which for an optimal filter $H(z)$ with $\sigma_n\sigma_\omega = 0.5$ yields $\text{tfu}(H) = 2$. Other numerical properties evaluated experimentally include the Frequency-Domain Selectivity $\text{fds}(H)$, Vanishing Moments Number $\text{vmn}(H)$, orthogonality and biorthogonality errors, filter bank reconstruction error, and others as described in the proposed standards [9, 10] for reproducibility of wavelet transform algorithms.

Filters were named with identifying acronyms followed by $(L_a, L_s; N_a, N_s)$ in the biorthogonal cases and by $(L; N)$ in the orthogonal cases. All results reported here were computed with Version 4.4b1 of WAVBOX Software [7]. A complete description of the filter design and analysis algorithms will be provided in the detailed final version [11] of this report.

3 Results

Daubechies Real Biorthogonal symmetric balanced Most Regular (DRBMR) filters were selected by maximizing $\text{tdr}(H)$ balanced between $A(z)$ and $S(z)$ subject to the constraint of lengths L_a and L_s also balanced with $L_a \approx L_s$ as much as possible. In fact, $L = L_a = L_s = 2N$ is possible with $N = N_a = N_s$ for $\{N = 1 + 4k \mid k = 1, 2, 3 \ldots\}$. Figure 1 and Table 1 present graphical and tabular results for DRBMR(10,10;5,5) which is the shortest in this family. Daubechies Real Orthogonal Least-Asymmetric (DROLA), Complex Orthogonal Least-Asymmetric (DCOLA), and Complex Orthogonal Most-Symmetric (DCOMS) filters were each selected by minimizing $\text{pnl}(H)$. Table 2 lists the coefficients for the filter DROLA(26;13). It has the value $\text{pnl}(H) = 0.32$, which is minimal for all $H(z)$ in the DROLA$(L; N)$ family with $L \le 40$, $N \le 20$. DROLA(26;13) is also the shortest in the family with $\text{tdr}(H) \ge 4$ and has an actual value of $\text{tdr}(H) = 4.07$; its other values include $\text{fds}(H) = 0.65$ and $\text{tfu}(H) = 3.41$. Table 2 also lists the coefficients for the filter DCOMS(22;11). Figure 2 displays the corresponding graphical results including the numerical values for $\text{pnl}(H)$, $\text{tdr}(H)$, $\text{fds}(H)$, and $\text{tfu}(H)$. Note that even though the coefficients are symmetric, the phase is neither linear nor symmetric as seen in the group delay $\text{gd}(H(\omega))$ plots. DCOMS(22;11) is the shortest in the DCOMS$(L; N)$ family with $\text{pnl}(H) \le 1$. Each of these examples passed their respective orthogonality or biorthogonality tests as well as their perfect reconstruction tests with relative errors ranging from $\mathcal{O}(10^{-17})$ to $\mathcal{O}(10^{-15})$.

4 Discussion

Explicit computational algorithms have been developed for generating Daubechies compact maximally flat wavelets with varying degrees of symmetry or asymmetry. The terms "symmetric" and "asymmetric" have been used in reference to the actual coefficients, whereas the modifying superlatives "least" and "most" have been used in reference to the phase nonlinearity of the coefficients as follows: asymmetric and symmetric coefficients with minimal phase nonlinearity ($\mathrm{pnl}(H) > 0$) have been called "least-asymmetric" and "most-symmetric" respectively for the real and complex orthogonal cases. The unmodified term "symmetric" has been used for the real biorthogonal case where symmetric coefficients with linear phase ($\mathrm{pnl}(H) = 0$) is possible.

Use of the automated algorithms results in the identification of new and interesting wavelets. Examples have been demonstrated for the DROLA, DRBMR, and DCOMS families (see Section 3). In particular, for the DRBMR family, an analysis-synthesis pair, each with $N = 5$ vanishing moments and length $L = 10$ coefficients, but with different time-domain regularities of $\mathrm{tdr}(A) = 1.213$ and $\mathrm{tdr}(S) = 2.321$, has been identified as the shortest of a sequence of pairs which occurs for $N = 1 + 4k$. This new biorthogonal (10,10) filter pair can be compared with the well-known (9,7) pair with regularities of 1.068 and 1.701. In the setting of image compression with symmetric biorthogonal filters, the increased regularity of the (10,10) pair should help reduce reconstruction artifacts.

These automated algorithms are valid for any order N of wavelet and insure that the same consistent choice of roots is always made in the computation of the filter coefficients. They are also sufficiently flexible for convenient generalization to the selection of roots for filters optimized for criteria other than those used here in this report. Complete tables of results for the systematized collection of wavelet filters based on a variety of optimization criteria and computable by spectral factorization of the Daubechies polynomial will be available in a forthcoming paper [11].

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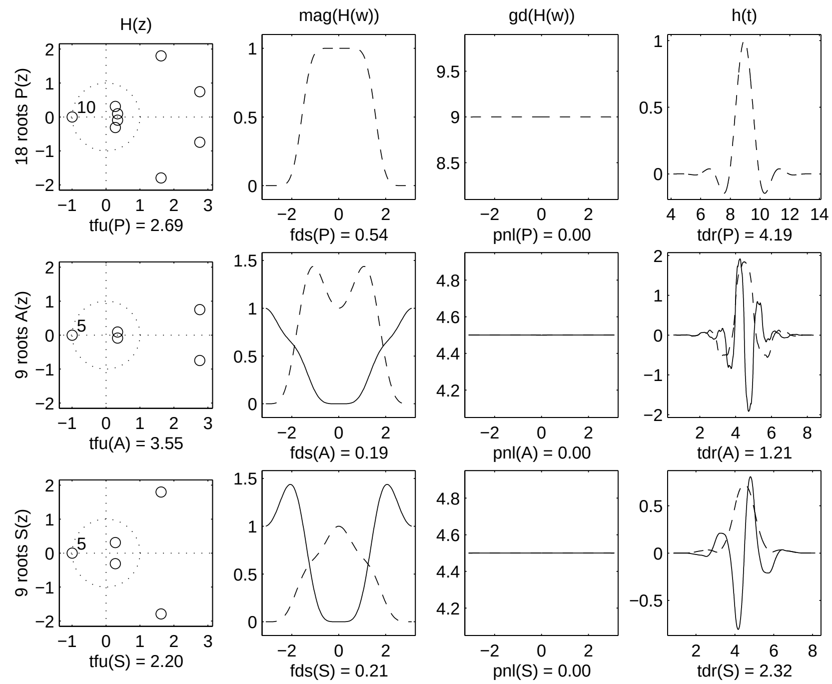

Figure 1: Daubechies Real Biorthogonal symmetric balanced Most Regular DRBMR(10,10;5,5).

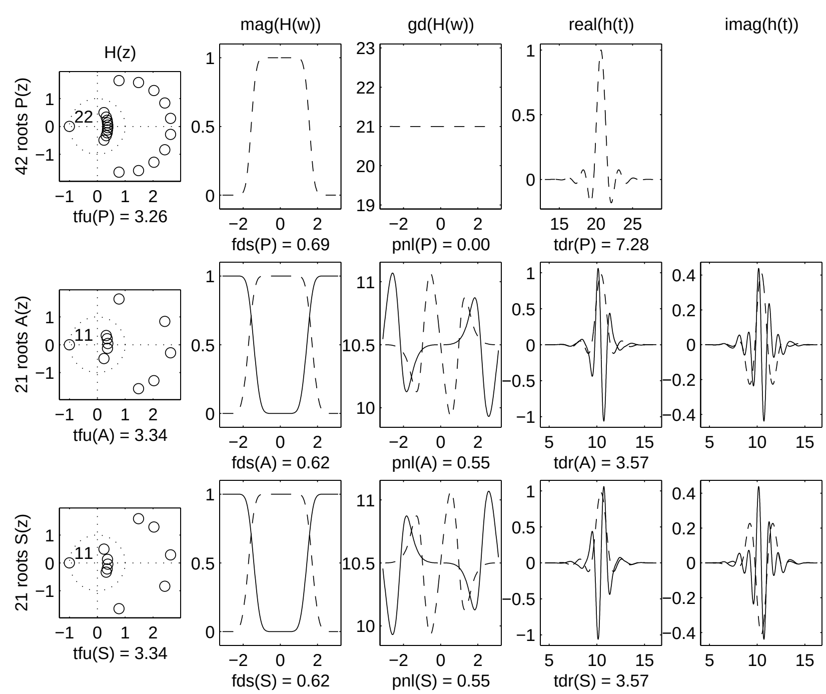

Figure 2: Daubechies Complex Orthogonal Most Symmetric DCOMS(22;11).

Table 1: Biorthogonal Symmetric Most-Regular Example

n	DRBMR(10,10;5,5)	
	analysis	synthesis
0	2.691341891883906e-2	1.984354417393126e-2
1	-3.230335268005382e-2	2.381759849262512e-2
2	-2.411098166478756e-1	-2.325783991548418e-2
3	5.410042184654684e-2	1.455707467437648e-1
4	8.995061097490912e-1	5.411327316917106e-1
5	8.995061097490908e-1	5.411327316917104e-1
6	5.410042184654706e-2	1.455707467437649e-1
7	-2.411098166478756e-1	-2.325783991548425e-2
8	-3.230335268005382e-2	2.381759849262512e-2
9	2.691341891883906e-2	1.984354417393126e-2

Table 2: Orthogonal Least-Asymmetric and Most-Symmetric Examples

n	DROLA(26;13)	DCOMS(22;11)	
		real part	imag part
0	7.042986690696272e-5	-2.843587691746741e-4	-5.619067497231089e-5
1	3.690537342323900e-5	3.943661457668242e-5	-1.995730997530724e-4
2	-7.213643851363769e-4	3.503067075329774e-3	-6.987157490609974e-6
3	4.132611988416842e-4	2.610443554801058e-4	1.162351282891239e-3
4	5.674853760123303e-3	-1.826877776716165e-2	3.762785603097234e-3
5	-1.492447274258628e-3	-2.986205433341184e-3	5.055616843377799e-3
6	-2.074968632552075e-2	5.069295483423355e-2	-1.130173637346818e-2
7	1.761829688064435e-2	-1.211437361767570e-2	-5.680614296894008e-2
8	9.292603089914724e-2	-1.204103902125976e-1	-7.712723996428508e-2
9	8.819757670420982e-3	1.474823589470471e-1	6.042637868194313e-3
10	-1.404900931136589e-1	6.591920251598299e-1	1.294744786413487e-1
11	1.102302230212641e-1	6.591920251598352e-1	1.294744786413491e-1
12	6.445643839011783e-1	1.474823589470424e-1	6.042637868193830e-3
13	6.957391505615532e-1	-1.204103902125941e-1	-7.712723996428485e-2
14	1.977048187712746e-1	-1.211437361767762e-2	-5.680614296894021e-2
15	-1.243624607515050e-1	5.069295483423437e-2	-1.130173637346811e-2
16	-5.975062771795706e-2	-2.986205433341443e-3	5.055616843377777e-3
17	1.386249743583920e-2	-1.826877776716159e-2	3.762785603097244e-3
18	-1.721164272630488e-2	2.610443554800936e-4	1.162351282891241e-3
19	-2.021676813339524e-2	3.503067075329776e-3	-6.987157490610306e-6
20	5.296359738721811e-3	3.943661457668191e-5	-1.995730997530723e-4
21	7.526225389968170e-3	-2.843587691746740e-4	-5.619067497231088e-5
22	-1.709428585295735e-4		
23	-1.136063438927966e-3		
24	-3.573862364871616e-5		
25	6.820325263074346e-5		